# Vibration Suppression of a Cart-Flexible Pole System Using a Hybrid Controller

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Abstract—A hybrid control strategy is used in this work to suppress the structural vibrations of a flexible system. The hybrid controller is based on the combination of inverse dynamics feedforward control, command shaping and linear state feedback control. The nonlinear feedforward control is derived using inverse dynamics, which is useful to linearize the system around the nominal trajectory. The feedback loop is designed with linear observer based optimal regulator which ensures stabilization and performance objectives. Finally, the command shaping is incorporated to obtain the desired nonoscillatory response. Command shaping is an effective way of improving the performance of systems with flexible dynamics, e.g. flexible manipulators, flexible structures, spacecraft with large appendages, ships, cranes and telescopes. The method is applied to the case of a flexible inverted pendulum on a moving cart. The simulation runs show the efficacy of the proposed controller in vibration suppression of a highly flexible system.

Keywords—Flexible manipulators; command shaping; vibration suppression; lumped parameter model; cart-pole

# I. INTRODUCTION

Faster response and energy efficient consumption are the two major factors, which have motivated the design of light-weight, flexible manipulators. Mostly, the conventional robots are designed with maximum stiffness to achieve good positional accuracy and non oscillatory response. The high stiffness leads to the bulky manipulators, which consumes more power and have low payload to robot weight ratio. The viable solution to such problems is to relax the stiffness constraint and seek flexible manipulators. These manipulators have lighter design, fast motion, low power consumption and high payload to robot weight ratio [1]. Research in the field of flexible manipulators started in 1970's when Book [2] initiated the research on modeling and control of the flexible link manipulators by modeling an elastic chain with an arbitrary number of links and joints. The initial studies [2]-[4] on the control of flexible manipulators started under the domain of space applications with the objective of minimizing the launching cost. Thereafter, flexible manipulators have been studied under various challenging applications like painting, drawing, polishing, pattern recognition, nuclear maintenance, storage tank cleaning and inspection, microsurgical operation, automated deburring and many other similar operations.

However, achieving precise control of such manipulators is a challenging task that is critical in many areas like nuclear industry, medical surgery and space applications. To obtain good positional accuracy, it is important to have a good mathematical model of the flexible system in hand. Several researches in the past have tried different modeling techniques to obtain the dynamic model of flexible manipulators. These modeling techniques are broadly classified into two main categories — distributed systems and discrete systems. Due to the infinite-dimensional model of distributed systems, solutions as well as controller design are more difficult as compared to the corresponding discrete models. Therefore, many researchers in the past have adopted approximate solutions to obtain discrete models of the flexible systems. These approximate solutions are broadly classified as lumped parameter methods (LPM) [5], assumed modes method (AMM) [6] and finite element method (FEM) [7].

Both feedforward as well as feedback control schemes have been separately used in the literature to control flexible manipulators. Feedforward strategy includes Fourier expansion, computed torque techniques, bang-bang control (openloop time-optimal control) and open-loop input shaping techniques. Bang-bang control requires two-impulse inputs, which excites all the modes of the structure and leads to high vibration levels. Open-loop input shaping is used extensively by researchers as an active vibration suppression strategy to obtain a non-oscillatory response.

# II. COMMAND SHAPING

Command shaping is an effective way of improving the performance of systems with flexible dynamics, e.g. flexible manipulators, flexible structures, spacecraft with large appendages, ships, cranes and telescopes. The method involves the convolution of the reference input with a sequence of impulses to obtain a non-oscillatory response. The input shaper variables, i.e. amplitudes and time instants of the impulses are found by satisfying a set of constraint equations which are functions of natural frequencies and damping ratios of the flexible system. However, following the nofree-lunch policy, commanded input signal comes with a



Fig. 1. Response of a LTI system to first impulse.



Fig. 2. Response of a LTI system to second impulse.



Fig. 3. Combined Response of a LTI system to both the impulses.

marginal time penalty — equal to the length (duration) of the shaper. Furthermore, the command shaping techniques are highly susceptible to modeling errors and parametric uncertainties. Therefore, it is assumed *a priori* that a well established dynamic model of the physical system exists.

#### A. Concept of Command Shaping

To introduce the concept, a simple case of response of the system to two impulses is taken. Impulse plays a pivotal role in all the command shaping techniques. Expected response of a second-order linear-time-invariant (LTI) system after being hit by an impulse of magnitude, say  $A_1$ , is shown in the Fig. 1. The impulse will cause the flexible system to vibrate with some frequency. To cancel the vibrations being induced in the system by the first impulse, a second impulse of magnitude  $A_2$  is applied on the system at a later stage. Figure 2 shows the response of the system to the second impulse. Using the superposition principle, both the responses can be combined together and their combination

will result in zero residual vibrations as shown in Fig. 3. However, the magnitude as well as timing of the second impulse must be very precise. This simple process shows that with the judicious use of impulses, it is possible to obtain vibration-free response.

## III. DYNAMIC MODEL

This section presents the dynamic model of a flexible manipulator (cart with an inverted flexible pendulum) using lumped parameter modeling. Using the Euler-Lagrange approach, the dynamic model of a general flexible manipulator (refer [8]) can be written in the standard form as

$$\mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{n}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) + \mathbf{g}(\boldsymbol{\theta}) + \mathbf{K}\boldsymbol{\theta} = \boldsymbol{\tau}, \quad (1)$$

where  $\boldsymbol{\theta} \in R^{n \times 1}$  is the vector of generalized coordinates,  $\mathbf{M}(\boldsymbol{\theta}) \in R^{n \times n}$  is the symmetric and positive-definite inertia matrix,  $\mathbf{n}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in R^{n \times 1}$  is the vector of Coriolis and centripetal forces,  $\mathbf{g}(\boldsymbol{\theta}) \in R^{n \times 1}$  is the vector of gravitational forces,  $\mathbf{K} \in R^{n \times n}$  is the diagonal stiffness matrix and  $\boldsymbol{\tau} \in R^{n \times 1}$  is the vector of generalized forces.

# A. Case Study: Cart with a Flexible Pole

In this section, dynamic model of the cart with a flexible pole is presented using the Euler-Lagrange approach, which is developed by the author in [9]. The flexible pole can be modeled as a series of rigid rods connected by torsional springs. For simplicity, the pole is assumed to consist of two such rigid segments interconnected by a torsional spring  $(k_2)$  as shown in the Fig. 4. For the cart-pole system, vector of generalized coordinates is given as  $\boldsymbol{\theta} = (x, \theta_1, \theta_2)^T$  and vector of generalized forces as  $\boldsymbol{\tau} = (f, 0, 0)$ . The dynamic model of the cart-pole system can be given as



Fig. 4. Moving cart with a flexible pole

$$\mathbf{M}(\boldsymbol{\theta}) = \begin{bmatrix} M + m_1 & (\frac{m_1}{2} + m_2)l_1C_1 & m_2\frac{l_2}{2}C_{12} \\ +m_2 & +m_2\frac{l_1}{2}C_{12} \end{bmatrix} \\ - & m_1\frac{l_1^2}{4} + m_2l_1^2 & m_2(\frac{l_2^2}{4}) \\ +m_2(\frac{l_2}{2} + l_1l_2C_2) & +m_2(l_1\frac{l_2}{2}C_2) \end{bmatrix} \\ - & - & m_2\frac{l_2^2}{4} \end{bmatrix}$$
(2)  
$$- & m_2\frac{l_2^2}{4} \end{bmatrix} \\ \mathbf{n}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} -(m_1\frac{l_1}{2}S_1 + m_2l_1S_1 + m_2l_1S_{12})\dot{\theta}_1^2 \\ -(m_2\frac{l_1}{2}S_{12} + m_2\frac{l_2}{2}S_{12})\dot{\theta}_1\dot{\theta}_2 \\ -m_2\frac{l_2}{2}S_{12}\theta_2^2 \end{bmatrix} \\ - & m_2(\frac{l_1-l_2}{2}S_{12})\dot{\theta}_1\dot{\theta}_2 \end{bmatrix} ,$$

$$\begin{bmatrix} -m_2(\frac{i_1-i_2}{2})S_{12}\dot{x}\theta_2\\ m_2l_1\frac{l_2}{2}S_2\dot{\theta}_1^2\\ -m_2(\frac{l_2-l_1}{2})S_{12}\dot{x}\dot{\theta}_1 \end{bmatrix}$$
(3)

$$\mathbf{g}(\boldsymbol{\theta}) = \begin{bmatrix} 0 \\ -(\frac{m_1}{2} + m_2)gl_1S_1 - m_2g\frac{l_2}{2}S_{12} \\ -m_2g\frac{l_2}{2}S_{12} \end{bmatrix}, \quad (4)$$
$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_1 & 0 \\ 0 & 0 & k_2 \end{bmatrix}. \quad (5)$$

This dynamic model has been utilized in the later section during the controller design phase of the cart-pole system.

# IV. LINEARIZATION OF THE DYNAMIC MODEL

The nonlinear dynamic model can be linearized about the nominal trajectory with  $\mathbf{P}(\boldsymbol{\tau}_n, \boldsymbol{\theta}_n, \dot{\boldsymbol{\theta}}_n, \ddot{\boldsymbol{\theta}}_n)$  as the nominal point. The general form of the linearized dynamic model [10] can be given as

$$\mathbf{M}_L \ddot{\mathbf{q}} + \mathbf{N}_L \dot{\mathbf{q}} + \mathbf{G}_L \mathbf{q} = \mathbf{u},\tag{6}$$

where  $\mathbf{M}_L, \mathbf{N}_L, \mathbf{G}_L$  are  $n \times n$  matrices, and other variables are defined as

$$\tau_{n} = \text{nominal torque}$$
  
=  $\mathbf{M}(\boldsymbol{\theta}_{n})\boldsymbol{\dot{\theta}_{n}} + \mathbf{n}(\boldsymbol{\theta}_{n},\boldsymbol{\dot{\theta}_{n}}) + \mathbf{g}(\boldsymbol{\theta}_{n}) + \mathbf{K}\boldsymbol{\theta}_{n},$  (7)

$$\boldsymbol{\theta}_n(t) = \text{nominal joint vector},$$
 (8)  
 $\mathbf{u} = \text{control-input vector}$  (9)

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$$\mathbf{u} = \text{control-input vector}, \tag{9}$$
$$\mathbf{q}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}_n(t) \tag{10}$$

$$\mathbf{q}(t) = \boldsymbol{\theta}(t) - \boldsymbol{\theta}_n(t) \tag{10}$$

deviation from the nominal trajectory. \_

The linearized dynamic model can be represented as

$$\mathbf{M}_{L} = \begin{bmatrix} M + m_{1} & (\frac{m_{1}}{2} + m_{2})l_{1} & m_{2}\frac{l_{2}}{2} \\ +m_{2} & +m_{2}\frac{l_{1}}{2} \\ & & \\ - & m_{1}\frac{l_{1}^{2}}{4} + m_{2}l_{1}^{2} & m_{2}(\frac{l_{2}^{2}}{4}) \\ & & +m_{2}(\frac{l_{2}^{2}}{4} + l_{1}l_{2}) & +m_{2}(l_{1}\frac{l_{2}}{2}) \\ & & \\ - & & - & m_{2}\frac{l_{2}^{2}}{4} \end{bmatrix},$$
(11)

$$\mathbf{N}_{L} = \begin{bmatrix} \frac{\partial \mathbf{n}}{\partial \dot{\boldsymbol{\theta}}} \end{bmatrix}_{P} = \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}, \qquad (12)$$

$$\mathbf{G}_{L} = \begin{bmatrix} \frac{\partial(\mathbf{n})}{\partial \theta} \end{bmatrix}_{P} + \begin{bmatrix} \frac{\partial(\mathbf{g})}{\partial \theta} \end{bmatrix}_{P} + \mathbf{K}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(\frac{m_{1}}{2} + m_{2})gl_{1} & -m_{2}g\frac{l_{2}}{2} \\ -m_{2}g\frac{l_{2}}{2} + k_{1} \\ 0 & -m_{2}g\frac{l_{2}}{2} & -m_{2}g\frac{l_{2}}{2} + k_{2} \end{bmatrix}$$
(13)

## A. State space model

The above linearized dynamic model can be represented in the standard state space form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{14}$$

where

$$\mathbf{x}(t) = \left\{ \begin{array}{c} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{array} \right\}, \quad \dot{\mathbf{x}}(t) = \left\{ \begin{array}{c} \dot{\mathbf{q}}(t) \\ \ddot{\mathbf{q}}(t) \end{array} \right\}, \quad (15)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_n & \mathbf{I}_n \\ & \\ -\mathbf{M}_L^{-1}\mathbf{G}_L & -\mathbf{M}_L^{-1}\mathbf{N}_L \end{bmatrix}, \mathbf{B} = \left\{ \begin{array}{c} \mathbf{0}_n \\ & \\ \mathbf{M}_L^{-1} \end{array} \right\}.$$

This state-space representation is helpful in checking the controllability and observability of the system. Moreover, it helps in designing the feedback loop of the controller, which is based on the linear control theory.

## B. Controllability and Observability

The controllability and observability test matrix for a linear, time-invariant (LTI) system is given by

$$\mathbf{R}_{c} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^{2}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}.$$
(16)  
$$\mathbf{R}_{o} = \begin{bmatrix} \mathbf{C}^{T} & \mathbf{A}^{T}\mathbf{C}^{T} & (\mathbf{A}^{T})^{2}\mathbf{C}^{T} & \cdots & (\mathbf{A}^{T})^{n-1}\mathbf{C}^{T} \end{bmatrix},$$
(17)

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#### V. CONTROLLER DESIGN

In this section, the procedure for controller design for the tracking problem of nonlinear, unstable flexible manipulators is presented. The control action of the considered controller is given as

$$\mathbf{u} = \mathbf{u}_{ff} + \mathbf{u}_{fb},\tag{18}$$

The controller consists of three parts: 1) a nonlinear feedforward term, 2) a linear observer based optimal feedback term and 3) an input shaper. The development of each part of the controller can be found at [9], where the error dynamics of the reduced-order compensator can be written as

$$\begin{cases} \dot{\mathbf{e}}(t) \\ \dot{\mathbf{e}}_{ou}(t) \end{cases} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K}_{opt} & -\mathbf{B}\mathbf{K}_{u} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \begin{cases} \mathbf{e}(t) \\ \mathbf{e}_{ou}(t) \end{cases} + \begin{cases} -\mathbf{A} \\ \mathbf{0} \end{cases} \mathbf{x}_{d}(t), \qquad (19)$$

where  $\mathbf{e}(t) = \mathbf{x}_d(t) - \mathbf{x}(t)$  is the tracking error,  $\mathbf{e}_{ou}(t) = \mathbf{x}_u(t) - \mathbf{x}_{ou}(t)$  is the estimation error,  $\mathbf{x}_d(t) = {\mathbf{q}_d^T, \dot{\mathbf{q}}_d^T}^T$  is the desired state-vector and  $\mathbf{K}_{opt}$  is the optimal **Geback** gain matrix being partitioned as  $\mathbf{K}_{opt} = [\mathbf{K}_m \ \mathbf{K}_u]$ . The feedback gain matrix  $\mathbf{K}_{opt}$  is obtained by exploiting the *optimal control theory*, which produces the best possible control system to achieve the desired performance objectives. The aim of the optimal control problem is to minimize the control energy and transient energy of the system by formulating the following objective function

$$J_{\infty}(t) = \int_{t}^{\infty} \left[ \mathbf{x}^{T}(\tau) \mathbf{Q}(\tau) \mathbf{x}(\tau) + \mathbf{u}^{T}(\tau) \mathbf{R}(\tau) \mathbf{u}(\tau) \right] d(\tau),$$
(20)

which is subjected to the following constraint

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t).$$
(21)

The outcome of the optimal control problem is the feedback gain matrix  $\mathbf{K}_{opt}$  such that the scalar function  $J_{\infty}(t)$  is minimized.

#### VI. SIMULATION RESULTS

This section demonstrates the efficacy of the proposed control scheme by implementing the controller on the cartpole system. The control objective is to move the cart by a required distance while preventing the flexible pendulum from falling. This nonlinear unstable system can be thought of as a robot showing the art of balancing a flexible stick on its palm.

# A. Case Study: Cart with a Flexible Pole

In the present section, the example of a moving cart with an inverted flexible pendulum is considered for the simulation. The control objective is to move the cart by one meter, while not letting the pendulum to fall, i.e. the desired statevector  $\mathbf{x}_d = \{1 \ 0 \ 0 \ 0 \ 0\}^T$ . For the simulation run, the numerical values of the parameters of the system are given in Table I. Putting these parameter values in the Eqns. (11), (12)

TABLE I. SYSTEM PARAMETERS OF THE CART-POLE SYSTEM.

Parameter	Units	Symbol	Value
Mass of the cart	kg	M	1
Mass of the first link	kg	$m_1$	0.05
Mass of the second link	kg	$m_2$	0.05
Length of the first link	m	$l_1$	1
Length of the second link	m	$l_2$	1
Spring stiffness at first joint	Nm/rad	$k_1$	5
Spring stiffness at second joint	Nm/rad	k2	500

TABLE II. CONTROLLER PARAMETERS OF THE CART-POLE SYSTEM.

Parameter	Symbol	Value
State weighting matrix	Q	diag([5000, 500, 0, 20, 0, 0])
Control cost matrix	R	[50]
Vector of desired ROO poles	v	$[-20, -22 \pm 60i, -60 \pm 69i]^T$
Optimal gain matrix	K <sub>opt</sub>	[10, -1.29, -0.43, 4.80, 0.02, -0.01]

TABLE III. SHAPER PARAMETERS OF THE CART-POLE SYSTEM.

Impulse	First Mode		Second Mode		Third Mode	
(i)	Magnitude	Time	Mag.	Time	Mag.	Time
1	0.9183	0.0000	0.2792	0.0000	0.250	0.0000
2	0.0799	1.4694	0.4984	0.5350	0.500	0.0049
3	0.0017	2.9387	0.2224	1.0701	0.250	0.0098

and (13),  $\mathbf{M}_L$ ,  $\mathbf{N}_L$  and  $\mathbf{G}_L$  are obtained, which are further used in Eqn. (14) to obtain the linear state-space representation. The only input to the system (u(t)) is the horizontal force applied to the cart and three outputs are the horizontal position of the cart (x(t)), angular position of the first link of the pendulum  $(\theta_1(t))$  and angular position of the second link of the pendulum  $(\theta_2(t))$ . The state-vector of this sixth-order plant is  $\mathbf{x}(t) = \left[x(t), \theta_1(t), \theta_2(t), \dot{x}(t), \dot{\theta}_1(t), \dot{\theta}_2(t)\right]^T$ .

First of all, controllability of the plant is found by using Eqn. (16). Rank of the controllability test matrix  $R_c$  is found to be 6, which implies that the plant is controllable. The eigenvalues of the plant are  $\lambda_{OL} =$  $[0, 0, \pm 5.89i, \pm 638.92i]$ . Due to the presence of a double pole at the origin, the plant is unstable to start with and cannot be controlled with the feedforward control only. Thus, feedback term  $\mathbf{u}_{fb}$  is necessarily required and is being calculated using the optimal control theory. The task of the feedback control is to stabilize the plant and obtain the positional accuracy.

The controller parameters are given in Table II. The selection of **Q** and **R** are made on the basis of trial and error to keep a check on the settling time, maximum overshoot and actuator effort. Using these parameters, optimal gain values are found as  $\mathbf{K}_{opt} = [10, -1.29, -0.43, 4.80, 0.02, -0.01]$ . With the selection of feedback gains, the eigenvalues of the closed-loop tracking system are found to be  $\lambda_{CL} = [-0.01 \pm 638.92i, -0.21\pm 5.87i, -2.13\pm 2.14i]$ . All the closed-loop eigenvalues with negative real parts justifies the selection of feedback gains and ensures asymptotic stability of the closed-loop system.

Out of the three outputs mentioned above, the reduced-



Fig. 5. Reference input vs shaped input to the cart-flexible-pole system.

order observer is designed by taking the cart's position as the only output variable. Thus, taking  $\mathbf{y}(t) = \{x(t)\}$ , the output coefficient matrices are given as  $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ and  $\mathbf{D} = \mathbf{0}$ . Using the position of the cart as the only output variable, observability test matrix  $\mathbf{R}_o$  is calculated using Eqn. (17). The rank of  $\mathbf{R}_o$  is found to be 6, which implies that the plant is observable with  $\mathbf{y}(t) = \{x(t)\}$ . The observer is designed by placing the observer poles suitably in the lefthalf plane, i.e.  $\mathbf{v} = \{-20, -20 \pm 60i, -60 \pm 69i\}^T$ . While designing the reduced-order observer, the first step is to partition the state-vector into measurable and unmeasurable parts i.e.  $\mathbf{x}(t) = \{\mathbf{x}_1(t)^T, \mathbf{x}_2(t)^T\}^T$ , where  $\mathbf{x}_1(t) = \{\mathbf{x}(t)\}$ and  $\mathbf{x}_2(t) = \{\theta_1(t), \theta_2(t), \dot{x}(t), \dot{\theta}_1(t), \dot{\theta}_2(t)\}^T$ . The output equation is  $\mathbf{y}(t) = \mathbf{C}\mathbf{x}_1(t)$ , where  $\mathbf{C} = 1$ . With the given choice of observer poles, the observer gain-matrix is found to be  $\mathbf{L} = \{-19.49, 193.59, 184, -10721.42, 36914.31\}^T$ . The shaper parameters for all the three modes are given in Table III. After getting the shaped input, the control action **u** is calculated using the Eqn. (18) and the dynamic model is solved to obtain the state-vector.

Figure 5 shows the reference unshaped input vs 1/2/3mode shaped input for the Cart in the x direction, whereas Fig. 6 shows the unshaped vs first-mode shaped response of the system. In this simulation, the unshaped response of the system is obtained using full-state-feedback-control (FSFC), whereas the shaped response is obtained using linear-optimal control theory along with a reduced-order observer. Figure 6-(a) shows the comparison of unshaped vs first-mode shaped response of the cart position in the horizontal direction, i.e. x(t). As mentioned earlier, the desired cart position  $x_d(t) =$ 1 meter. It can be observed that in the case of unshaped response, the extreme right position of the cart is 1.0614 meter, which is reduced to 1.0112 meter in the case of firstmode shaped response. This results in a 4.73% reduction in the extreme cart position, i.e.  $|x(t)_{max}|$ . Similar trend can be observed in peak vibration amplitudes of  $\theta_1(t)$ ,  $\theta_2(t)$ and u(t), as shown in the Fig. 6-(b), (c) and (d) respectively, where a reduction of 8.11%, 8.32% and 10.79% is found in the shaped response and control input.

In the next simulation, response of the system to two-



Fig. 6. Unshaped vs first-mode shaped position response and required control-input plot of the cart-flexible-pole system.

mode shaping process is observed. The desired position in this case is obtained by the convolution of first two modes. It can be noticed in Fig. 7-(a) that  $|x(t)_{max}|$  is reduced to 1 meter in the two-mode shaped response as compared to 1.0614 meter in the case of unshaped response, resulting in a 5.78% reduction in the overshoot of cart position. Similarly, as shown in the Figs. 7-(b) and 7-(c), a significant reduction is observed in the peak vibration amplitudes of  $\theta_1(t)$  and  $\theta_2(t)$  respectively. To be precise, there is a 80.99% reduction in the peak vibration amplitude of  $\theta_1(t)$  and 80.01% reduction in  $\theta_2(t)$ .

It can also be seen in Fig. 7-(b) that the vibrations present in the unshaped position of the first link of the flexible pendulum quickly decays in the shaped response, whereas there are still some vibrations left in the shaped response of  $\theta_2(t)$ . These vibrations of the second link show the structural vibrations of the flexible pendulum, which are yet to be taken into account in the shaping process. The amount of control input has also reduced by 56.83%, as shown in the Fig. 7-(d). It is worth mentioning here that the time period of the second mode is 1.0701 second. Thus, the two-mode shaped response is delayed by this much



Fig. 7. Unshaped vs two-mode shaped position response and required control-input plot of the cart-flexible-pole system.



Fig. 8. Unshaped vs two-mode shaped velocity response and control-input contribution plot of the cart-flexible-pole system.

Variable	Unshaped	One Mode Shaping		Two Mode Shaping		Three Mode Shaping	
	Response	Value	% Reduction	Value	% Reduction	Value	% Reduction
x(t) <sub>max</sub>	1.0614	1.0112	4.73 %	1.0000	5.78 %	1.0000	5.78 %
$ \theta_1(t)_{max} $	13.0915	12.030	8.11 %	2.488	80.99 %	2.488	81.00 %
$ \theta_2(t)_{max} $	0.0429	0.0393	8.32 %	0.0086	80.01 %	0.0078	81.83 %
dx(t) <sub>max</sub>	1.3069	1.2002	8.17 %	0.7903	39.53 %	0.7901	39.54 %
$\left d\theta_{1}(t)_{max}\right $	1.1555	1.0613	8.16 %	0.2349	79.67 %	0.2272	80.33 %
$\left d\theta_{2}(t)_{max}\right $	0.0616	0.0585	10.05 %	0.0148	75.97 %	0.0089	86.36 %
u(t) <sub>max</sub>	10.00	8.9206	10.79 %	4.3165	56.83 %	4.1920	58.08 %

TABLE IV. LEVEL OF VIBRATION REDUCTION AND CONTROL TORQUE COMPARISON OF UNSHAPED VS SHAPED RESPONSE.

amount as compared to the one-mode shaped response. The comparison of rate responses of the state-variables was also shown in Fig. 8, where a similar trend can be observed. A significant reduction of 39.53%, 79.67% and 75.97% is observed in the peak vibration amplitudes of  $\dot{x}(t)$ ,  $\dot{\theta}_1(t)$  and  $\dot{\theta}_2(t)$  respectively.

In the last simulation run under this category, all the three modes are convolved together to modify the desired profile of the cart position  $x_d(t)$ . The comparison of unshaped profile with one, two and three mode shaped profile is given in the Fig. 5. This comparison demonstrates how the desired profile gets modified with the use of multi-mode shaping. Furthermore, the additional delay can be seen in achieving the desired cart position of 1 meter. As shown in the above figure, there is hardly any difference in two-mode and three-mode shaped profile. This is due to the fact that the stiffness of the second spring  $k_2$  is kept very high to introduce high-frequency vibrations into the system. These high-frequency vibrations represent structural vibrations of the flexible pendulum. Due to very high frequency, the time period of the third mode is 0.0098 seconds. This means that the response of the system to three-mode shaped profile is delayed by 0.0098 seconds as compared to the response of the system to two-mode shaped profile.



Fig. 9. Unshaped vs three-mode shaped position response and required control-input plot of the cart-flexible-pole system.

It can be clearly seen in Fig. 9-(a) that similar to the twomode shaping case, there is a 5.78% reduction in  $|x(t)_{max}|$ . Further, reduction in the peak vibration amplitude of  $\theta_1$ and  $\theta_2$  has increased marginally to 81% and 81.83%, as shown in Figs. 9-(b) and 9-(c), respectively. The reduction in the control input has increased to 58.08%, as shown in Fig. 9-(d). All the results of unshaped vs 1/2/3-mode shaped response are tabulated in Table IV. As the third mode depicts the structural vibrations of the flexible pendulum, significant reductions are observed in  $\theta_2$  and  $\dot{\theta}_2$ , whereas there are only marginal reductions in the remaining state-variables. Explicitly, there is an additional reduction of 1.82% in  $|\theta_2(t)_{max}|$ and 10.39% in  $|\dot{\theta}_2(t)_{max}|$ , as compared to the two-mode shaped response. Hence, all the three simulation runs clearly demonstrates the efficacy of the proposed control scheme in significantly reducing the vibration levels as well as control input to the unstable plant.

## VII. CONCLUSIONS

This paper has presented the development of a hybrid controller for flexible manipulators, which is a combination of nonlinear feedforward term, linear state feedback term and command shaping technique. It is shown in this work that linear feedback is sufficient to control a highly nonlinear and unstable flexible system. This work has presented the implementation of a linear observer based feedback strategy on the cart-pole system. The control objective was to move the cart by a required distance while not letting the flexible pendulum to fall. A comparison of the results has demonstrated that the controller designed for the shaped input is working better than the one designed for unshaped input. Effectiveness of the proposed controller has been demonstrated by showing a significant reduction in the vibration levels as well as in the actuator effort. Also, the importance of command shaping scheme has been demonstrated by achieving a nonoscillatory response.

#### APPENDIX

## A. Zero Vibration/Two-Impulse Sequence

In the last section, the analytical development of the shaper is presented. It is already shown in Fig. 3 that with the sensible use of two impulses, vibration-free response can be obtained. These, first shapers of their generation, can be formed using constraints which can limit the residual vibration of the system to zero at the modeled natural frequency and damping ratio, provided there is no error in the modeled frequency. Thus, these shapers are typically known as *Zero Vibration* (ZV) shapers [11]. In this section, the analytical method to find the amplitudes  $(A_i)$  and time locations  $(t_i)$  of ZV shapers is outlined. It is well known that the behavior of an *n*-th order system can be well represented by the superposition of second-order systems. The transfer function of such a general, underdamped, second-order system can be given as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} , \qquad (22)$$

where  $\omega_n$  and  $\zeta$  are the natural frequency and damping ratio of the underdamped system respectively. The impulse response of a second-order system, which is valid for  $0 < \zeta < 1$ , is given as

$$y_{o}(t) = \frac{A_{o}\omega_{n}}{\sqrt{1-\zeta^{2}}}e^{-\zeta\omega_{n}(t-t_{o})}\sin(\omega_{n}\sqrt{1-\zeta^{2}}(t-t_{o})),$$
(23)

where  $A_o$  and  $t_o$  are amplitude and time instant, when the impulse is applied. Using the principle of superposition, response of the system to a sequence of N impulses after the time of last impulse can be obtained as

$$y(t) = \sum_{i=1}^{N} \left[ \frac{A_i \omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n (t-t_i)} \right] \sin(\omega_n \sqrt{1-\zeta^2} (t-t_i)),$$
(24)

where  $A_i$  and  $t_i$  are the amplitudes and time instants of the *i*-th impulse. Using polar coordinates, the above equation can be written in more compact form as

$$y(t) = P\sin(\omega_d t + \beta), \qquad (25)$$

The above solution is valid for  $t > t_N$ . Here,  $\beta$  represents the phase shift (unimportant here) and P is the residual vibration amplitude given as

$$P = \sqrt{\left(\sum_{i=1}^{N} P_i \cos(\omega_d t_i)\right)^2 + \left(\sum_{i=1}^{N} P_i \sin(\omega_d t_i)\right)^2}, \quad (26)$$

where

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = \text{Damped natural frequency,(27)}$$
  

$$P_i = \frac{A_i \omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n (t - t_i)}.$$
(28)

To calculate P caused by an impulse sequence of N impulses, (26) is evaluated at the time instant of last impulse,  $t = t_N$ . Substituting the value of  $P_i$  from (28) in (26), and taking constant terms out of the square root, the amplitude of single-mode residual vibrations (refer [1], [12], [13]), as an outcome to a sequence of impulses, can be obtained as

$$P = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_N} \sqrt{Q_1 + Q_2}, \qquad (29)$$
  
where  $Q_1 = \left\{ \sum_{i=1}^N A_i e^{\zeta\omega_n t_i} \cos(\omega_d t_i) \right\}^2,$   
 $Q_2 = \left\{ \sum_{i=1}^N A_i e^{\zeta\omega_n t_i} \sin(\omega_d t_i) \right\}^2.$ 

To represent the above amplitude as a non-dimensional quantity (refer [14]), (30) can be divided by the amplitude of a residual vibration from a single impulse of a unity magnitude  $(P_{\delta})$ , which is given as

$$P_{\delta} = \frac{\omega_n}{\sqrt{1-\zeta^2}}.$$
(30)

The resulting percentage of residual vibration expression i.e.  $V(\omega_n, \zeta) = \frac{P}{P_{\delta}}$ , represents the ratio of vibration with input shaping to that without input shaping. Thus, the relative vibration expression (in percentage values) can be written as

$$V(\omega_{n},\zeta) = e^{-\zeta\omega_{n}t_{N}} \sqrt{\left[V_{C}(\omega_{n},\zeta)\right]^{2} + \left[V_{S}(\omega_{n},\zeta)\right]^{2}},$$

$$V_{C}(\omega_{n},\zeta) = \sum_{i=1}^{N} A_{i}e^{\zeta\omega_{n}t_{i}}\cos(\omega_{d}t_{i}),$$

$$V_{S}(\omega_{n},\zeta) = \sum_{i=1}^{N} A_{i}e^{\zeta\omega_{n}t_{i}}\sin(\omega_{d}t_{i}).$$
(31)

The zero-vibration solution leads us to set  $V(\omega_n, \zeta) = 0$ and solve for the variables of the shaper. However, since the Eqn. (31) is nonlinear and under-determined in nature, infinite possible solutions can exist. To avoid trivial solutions, i.e. zero-valued and infinite-valued impulses [15], it is better to put some constraints on the system. Thus, the two-impulse sequence can be found by solving a nonlinear system of equations as

$$V(\omega_n,\zeta) = 0, \tag{32}$$

$$\sum_{i=1} A_i = 1, \tag{33}$$

$$A_i > 0 \quad \forall \ i. \tag{34}$$

The above problem has four variables  $(A_1, A_2, t_1, t_2)$ . The second equation, i.e. Eqn. (33) signifies that the shaped command signal produces the same rigid-body motion as that of the reference signal. The next inequality in (34) avoids saturation of the actuators. To avoid more delay, choose

 $t_1 = 0$ , leading to a nonlinear system with three unknowns only. Thus, Eqn. (31) can be written in an expanded form as

$$V_C(\omega_n, \zeta) = \sum_{i=1}^2 A_i e^{\zeta \omega_n t_i} \cos(\omega_d t_i)$$
(35)  
$$= A_1 e^{\zeta \omega_n t_1} \cos(\omega_d t_1) + A_2 e^{\zeta \omega_n t_2} \cos(\omega_d t_2)$$
  
$$= 0,$$

$$V_{S}(\omega_{n},\zeta) = \sum_{i=1}^{2} A_{i} e^{\zeta \omega_{n} t_{i}} \sin(\omega_{d} t_{i})$$
(36)  
$$= A_{1} e^{\zeta \omega_{n} t_{1}} \sin(\omega_{d} t_{1}) + A_{2} e^{\zeta \omega_{n} t_{2}} \sin(\omega_{d} t_{2})$$
  
$$= 0.$$

Using  $t_1 = 0$ , the above equations can be written as

$$A_1 + A_2 e^{\zeta \omega_n t_2} \cos(\omega_d t_2) = 0,$$
 (37)

$$A_2 e^{\zeta \omega_n t_2} \sin(\omega_d t_2) = 0. \tag{38}$$

Solving above equations, the ZV shaper in compact form can be written as

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{t} \end{bmatrix}_{2} = \begin{bmatrix} \frac{1}{1+M} & \frac{M}{1+M} \\ 0 & \frac{\pi}{\omega_{d}} \end{bmatrix}, \quad (39)$$

where 
$$M = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}}$$
. (40)

The ZV shaper is highly susceptible to modeling errors and parametric uncertainties. That is why it is being called as a non-robust shaper. To tackle this problem, robustness issue as well as more robust shapers are presented in next sections.

#### B. Robustness of an Input Shaper

As mentioned earlier, the input shaper variables, i.e. amplitudes and time locations (instants) of the impulses are functions of natural frequencies and damping ratios of the physical system. The success of input shaping technique depends purely on the accuracy of the mathematical model of the physical system. Due to parametric uncertainties and modeling inaccuracies, deriving an accurate mathematical model is quite challenging. In such a scenario, input shaping will not produce a zero vibration response. The input shaper can be made insensitive to errors in natural frequencies of the system by setting  $\frac{\partial V(\omega_n,\zeta)}{\partial \omega_v} = 0$ . Solving for the shaper parameters after the addition of this constraint results in a three-impulse sequence, which is also known as *Zero Vibration and Derivative* (ZVD) shaper. The ZVD shaper [16] is given as

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{t} \end{bmatrix}_{3} = \begin{bmatrix} \frac{1}{1+2M+M^{2}} & \frac{2M}{1+2M+M^{2}} & \frac{M^{2}}{1+2M+M^{2}} \\ 0 & \frac{\pi}{\omega_{d}} & \frac{2\pi}{\omega_{d}} \end{bmatrix}.$$
(41)

This shaper is typically used where a fair amount of uncertainty can not be neglected in the system parameters. It can be seen that the increase in robustness of ZVD shaper is earned at the cost of a marginal time penalty, which increases the time-lag of the system. The length (duration) of the ZVD shaper is exactly twice that of the ZV shaper.

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