Dynamics of a Spatial Multibody System using Equimomental System of Point-masses

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***Abstract*—This paper presents dynamic modelling of a spatial multibody system using the equimomental system of point-masses which has several advantages, e.g., one needs to write only the translational equations of motion without the necessity of writing the rotational equations as the point masses have no dimension. For this, a rigid body is represented as a set of rigidly connected seven point-masses. Accordingly, the velocities of the point-masses are derived as a linear transformation of the joint-rates resulting into a set of three decoupled matrices called the Decoupled Natural Orthogonal Complement (DeNOC) matrices for the point-masses. The matrices are then used to form a minimum set of constrained equations of motion from the uncoupled Newton's equations of linear motion. The methodology is illustrated using a spatial solid pendulum.**

Keywords—Dynamics, spatial multibody system, equimomental system, DeNOC.

I. INTRODUCTION

Dealing with the dynamics of a spatial multibody system is always a challenging task. Dynamic equations of motion of the multibody system can be written using various coordinate systems. Absolute coordinate system represents the configuration of a body relative to the global reference frame. This results in a large number of equations of motion [1]. In joint coordinate system, the generalized coordinates are the relative coordinates between the bodies connected by a kinematic pair or joint [2]. On the other hand, an equimomental system of point masses is the one which consists of several point masses whose total mass, overall mass-center location and the moment of inertia are same as that of the original rigid body [3-4]. Such representation has several advantages. For example, one needs to deal with the translational equations of motion only, as the point masses have no dimension. Besides, it is more convenient to perform optimum design of the shapes of the bodies forming the overall multibody system, and its balancing. These advantages have motivated the researchers to use the concept of "Natural" coordinates", as presented in [5] and others. In [6] point coordinates replaced the rigid body and the equations of motion were derived in joint coordinates using the velocity transformation matrix. In [7-8] point coordinates is used to represent the rigid body and linear and angular momentum concept applied for dynamical formulation. The point mass representation was also used in [9-11], where balancing of several mechanisms was carried out.

This paper presents the modelling of a spatial multibody system using the concept of equimomental system of point-masses and the corresponding Decoupled Natural Orthogonal Complement (DeNOC) matrices. A set of seven point-masses, as proposed in [11], was used to define a rigid body moving in the three-dimensional Cartesian space. The preference of seven point-masses over typically used four point-masses [3] is mainly due to the presence of linear set of algebraic equations to find the point-masses [11]. The point-mass velocities were then expressed as a linear transformation of the joint-rates. The associated matrix is called the DeNOC matrices for the point-mass system, similar to the one presented in [12-13] for a serial-chain system. Further the equations of motion were derived using the Newton's equations of linear motion only. The methodology is shown by applying it to a spatial solid pendulum. A similar methodology was presented in [14] for a planar system.

The paper is organized as follows: Section II introduces the concept of point-mass system for a rigid-body in a spatial motion, whereas Section III derives the DeNOC matrices for the point-mass system. Section IV presents dynamic modeling and Section V illustrates the modeling using a spatial pendulum. Finally, Section 6 concludes the paper.

II. EQUIMOMENTAL SYSTEM OF SEVEN POINT-MASSES

Consider a rigid body or link, denoted as i^{th} link, moving in three-dimensional Cartesian space. The rigid link is dynamically represented as a set of rigidly connected seven point-masses, as proposed in [10-11]. It is shown in the Fig.1. To avoid the coincidence of any two points, the point-masses m_{ij} are located at the vertices of a parallelepiped, whose center is located at the origin O_{i+1} which has sides $2h_{ix}, 2h_{iy}, 2h_{iz}$ that are parallel to the axes of the body-fixed frame $O_{i+1}X_{i+1}Y_{i+1}Z_{i+1}$. The two systems of rigid-body and seven point-mass system are equimomental if the following conditions are satisfied:

$$m_{i1} + m_{i2} + m_{i3} + m_{i4} + m_{i5} + m_{i6} + m_{i7} = m_i \qquad (1)$$

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$$(m_{i1}+m_{i2}-m_{i3}-m_{i4}-m_{i5}+m_{i6}+m_{i7})h_{ix}=m_i\bar{x}_i \qquad (2)$$

$$(m_{i1}+m_{i2}+m_{i3}+m_{i4}-m_{i5}-m_{i6}-m_{i7})h_{iy}=m_i\bar{y}_i \qquad (3)$$

$$(m_{i1}-m_{i2}-m_{i3}+m_{i4}+m_{i5}+m_{i6}-m_{i7})h_{iz}=m_i\bar{z}_i \qquad (4)$$

$$(m_{i1}+m_{i2}-m_{i3}-m_{i4}+m_{i5}-m_{i6}-m_{i7})h_{ix}h_{iy}=I_{i,xy} \qquad (5)$$

$$(m_{i1}-m_{i2}-m_{i3}+m_{i4}-m_{i5}-m_{i6}+m_{i7})h_{iy}h_{iz}=I_{i,yz} \qquad (6)$$

$$(m_{i1}-m_{i2}+m_{i3}-m_{i4}-m_{i5}+m_{i6}-m_{i7})h_{iz}h_{ix}=I_{i,zx} \qquad (7)$$

$$\sum_{j=1}^{7} m_{ij}(h_{iy}^2+h_{iz}^2)=I_{i,xx} \qquad (8)$$

$$\sum_{j=1}^{7} m_{ij}(h_{iz}^2+h_{ix}^2)=I_{i,yy} \qquad (9)$$

$$\sum_{j=1}^{7} m_{ij}(h_{ix}^2+h_{iy}^2)=I_{i,zz} \qquad (10)$$

In (1-10), m_i is mass, $(\bar{x}_i,\bar{y}_i,\bar{z}_i)$ is mass-center location, $(I_{i,xx},I_{i,yy},I_{i,zz})$ are the moment of inertia, and $(I_{i,xy},I_{i,yz},I_{i,zx})$ are the product of inertia of the link at hand. The mass-center and the inertia tensor are referred in the body-fixed frame denoted as $O_{i+1}X_{i+1}Y_{i+1}Z_{i+1}$. Equation in (1) ensures the mass of the equimomental system is same as that of the rigid body, whereas those in (2-4) represent the same mass-center locations, and in (5-10) ensure the same inertia tensors about point O_{i+1}. Note that (2-10) are nonlinear in parameters, m_{ij}, h_{ix}, h_{iy} and h_{iz}. The parameters h_{ix}, h_{iy} and h_{iz} were found separately using (8-10), as proposed in [10-11]. This is done in the following way:

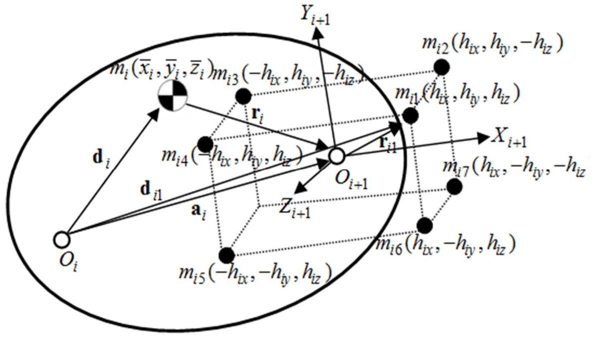


Fig. 1. Seven point-mass system

$$h_{ix}^2=(-I_{i,xx}+I_{i,yy}+I_{i,zz})/(2m_i) \qquad (11)$$

$$h_{iy}^2=(I_{i,xx}-I_{i,yy}+I_{i,zz})/(2m_i) \qquad (12)$$

$$h_{iz}^2=(I_{i,xx}+I_{i,yy}-I_{i,zz})/(2m_i) \qquad (13)$$

Note that the sum of any two moment of inertias to be greater than the third one [3]. Hence, h_{ix}, h_{iy} and h_{iz} will never have imaginary values. Then (2-7) becomes linear in $m_{i1},...,m_{17}$ and solve for them. This will be carried out in Section V.

III. DeNOC Matrices for the Point-mass System

Consider the i^{th} rigid link in spatial motion. The link is represented as a set of rigidly connected seven point-masses shown in Fig. 1. The velocities of the point-masses of the i^{th} link is derived in general form from the velocity of the origin O_i of the link as

$$\begin{aligned}\mathbf{v}_{i1} &= \mathbf{v}_i+\boldsymbol{\omega}_i\times\mathbf{d}_{i1}\\ &\vdots\\ \mathbf{v}_{i7} &= \mathbf{v}_i+\boldsymbol{\omega}_i\times\mathbf{d}_{i7}\end{aligned} \qquad (14\text{-}20)$$

where $\boldsymbol{\omega}_i$ and $\mathbf{v}_i$ are the angular velocity and linear velocity of the origin O_i of the i^{th} link, respectively, whereas the vector $\mathbf{d}_{ij}=\mathbf{a}_i+\mathbf{r}_{ij}$, for $j=1,\cdots,7$, denotes the position of the j^{th} point-mass of the i^{th} body from its origin O_i. Vector $\mathbf{a}_i$ is the link length vector of the i^{th} body, i.e., the vector representing the position of O_{i+1} from O_i, whereas vector $\mathbf{r}_{ij}$, $j=1,...,7$, is the position of the j^{th} point-mass from O_{i+1}. For a system of seven point-masses, their velocities are given by (14-20), which can be expressed in a compact form as

$$\tilde{\mathbf{v}}_i=\mathbf{D}_i\mathbf{t}_i \qquad (21)$$

where $\tilde{\mathbf{v}}_i$ is the 21-dimensional vector of point-mass velocities, and $\mathbf{D}_i$ is the 21×6 point-mass matrix defined as

$$\mathbf{D}_i \equiv \begin{bmatrix}-\mathbf{d}_{i1}\times\mathbf{1} & \mathbf{1}\\ \vdots & \vdots\\ -\mathbf{d}_{i7}\times\mathbf{1} & \mathbf{1}\end{bmatrix} \qquad (22)$$

In (22), $\mathbf{d}_{ij}\times\mathbf{1}$, for $j=1,...,7$, is the 3×3 cross-product tensor associated with the vector $\mathbf{d}_{ij}$, i.e., $(\mathbf{d}_{ij}\times\mathbf{1})\mathbf{x}=\mathbf{d}_{ij}\times\mathbf{x}$, for any 3-dimensional Cartesian vector $\mathbf{x}$. Moreover, the term $\mathbf{1}$ is the 3×3 identity matrix, and $\mathbf{t}_i$ is defined as the twist of the i^{th} body [11, 13]. For a rigid body moving in the three-dimensional Cartesian space, the 6-dimensional twist is defined as

$$\mathbf{t}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \\ \mathbf{v}_i \end{bmatrix} \quad (23)$$

where $\boldsymbol{\omega}_i$ and $\mathbf{v}_i$ are the 3-dimensional vectors of angular velocity and linear velocity of the origin of the i^{th} body, i.e., O_i, respectively. If the rigid bodies are coupled with the help of a single-degree-of-freedom (DoF) joints, say, a revolute and prismatic, the twists of all rigid bodies in the system represented as a generalized twist, i.e., $\mathbf{t} \equiv \begin{bmatrix} \mathbf{t}_1^T & \cdots & \mathbf{t}_n^T \end{bmatrix}^T$ can be written as

$$\mathbf{t} = \mathbf{N}\dot{\boldsymbol{\theta}}, \text{ where } \dot{\boldsymbol{\theta}} \equiv \begin{bmatrix} \dot{\theta}_1 & \cdots & \dot{\theta}_n \end{bmatrix}^T \quad (24)$$

and the matrix $\mathbf{N}$ is referred to as the Natural Orthogonal Complement or NOC of the velocity constraint matrix [13]. However, if the bodies or the links are connected with a joint having more than one-DoF joint, say, a 3-DoF spherical joint, then the scalar $\dot{\theta}_i$ associated with the i^{th} joint needs to be replaced with the 3-DoF joint-rate vector. Now, if a multibody system has serial-chain with n rigid bodies, the velocities of its equimomental point-masses can be expressed as

$$\tilde{\mathbf{v}} = \mathbf{D}\mathbf{t} \quad (25)$$

where $\tilde{\mathbf{v}}$ is the $21n$-dimensional vector, $\mathbf{t}$ is the $6n$-dimensional vector of generalized twist, and $\mathbf{D}$ is $21n \times 6n$ matrix defined as

$$\mathbf{D} \equiv diag.\left[\mathbf{D}_1, \ldots, \mathbf{D}_n\right] \quad (26)$$

Combining (24-26), one can write the point-mass velocities $\tilde{\mathbf{v}}$, in terms of the joint rates, i,e., $\dot{\boldsymbol{\theta}}$, as

$$\tilde{\mathbf{v}} = \tilde{\mathbf{N}}\dot{\boldsymbol{\theta}}, \text{ where } \tilde{\mathbf{N}} \equiv \mathbf{D}\mathbf{N} \text{ and } \mathbf{N} \equiv \mathbf{N}_l\mathbf{N}_d \quad (27)$$

The matrices $\mathbf{N}_l$ and $\mathbf{N}_d$ for one-DOF joints coupling the bodies are given in [13], which are given by

$$\mathbf{N}_l \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{A}_{21} & \mathbf{1} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \cdots & \mathbf{1} \end{bmatrix}; \; \mathbf{N}_d \equiv \begin{bmatrix} \mathbf{p}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{p}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{p}_n \end{bmatrix} \quad (28)$$

Where $\mathbf{O}$ is the matrix of zeros whose dimension is compatible according to the matrix where it appears. For example, $\mathbf{O}$ of $\mathbf{N}_l$ is the 6×6 matrix of zeros. Similarly, in $\mathbf{N}_d$ of (28), $\mathbf{0}$ represents the 6-dimensional vector of zeros. Moreover, $\mathbf{1}$ denotes the 3×3 identity matrix, and the 6×6 matrices of $\mathbf{N}_l$, i.e., $\mathbf{A}_{i,i-1}$, called twist propagation matrix. Finally, the vectors of $\mathbf{N}_d$, i.e., $\mathbf{p}_i$, are the joint-motion propagation vector. The expressions for the 6×6 matrix, $\mathbf{A}_{i,i-1}$, and the 6-dimensional vector $\mathbf{p}_i$ are given by

$$\mathbf{A}_{i,i-1} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{a}_{i,i-1} \times \mathbf{1} & \mathbf{1} \end{bmatrix}; \mathbf{p}_i \equiv \begin{bmatrix} \mathbf{e}_i \\ \mathbf{0} \end{bmatrix} \quad (29)$$

in which $\mathbf{a}_{i,i-1} = -\mathbf{a}_{i-1}$ is the vector associated to the skew-symmetric matrix $\mathbf{a}_{i,i-1} \times \mathbf{1}$. Note here that if the joints are having more DoF, the joint-motion propagation vector becomes a matrix whose columns are corresponding to the DoF of the joint at hand. For a 3-DoF spherical joint, illustration will be given in Section V.

IV. DYNAMIC MODELLING

The equations of motion for the point-mass system were derived using Newton's equations of linear motion and the corresponding DeNOC matrices derived in Section III. The unconstrained Newton's equations of motion for the jth point-mass of the ith body are given by

$$m_{ij}\dot{\mathbf{v}}_{ij} = \mathbf{f}_{ij}, \text{ for } j = 1, \ldots, 7 \quad (30)$$

where m_{ij} is the mass of the j^{th} point-mass in the i^{th} body, whereas, $\dot{\mathbf{v}}_{ij}$ is the acceleration of point-mass, and $\mathbf{f}_{ij}$ is the force including gravity acting on the j^{th} point mass of the i^{th} body. Equation in (30) can be written for an n-link system as

$$\tilde{\mathbf{M}}\dot{\tilde{\mathbf{v}}} = \tilde{\mathbf{f}} \quad (31)$$

In (31), $21n \times 21n$ matrix $\tilde{\mathbf{M}}$ and the $21n$-dimensional vectors $\dot{\tilde{\mathbf{v}}}$ and $\tilde{\mathbf{f}}$ are defined as follows:

$$\tilde{\mathbf{M}} \equiv diag.\left[m_{11}\mathbf{1} \ldots m_{17}\mathbf{1}, \ldots, m_{n1}\mathbf{1} \ldots m_{n7}\mathbf{1}\right] \quad (32a)$$

$$\dot{\tilde{\mathbf{v}}} = \left[\dot{\mathbf{v}}_{11}^T \ldots \dot{\mathbf{v}}_{17}^T, \ldots, \dot{\mathbf{v}}_{n1}^T \ldots \dot{\mathbf{v}}_{n7}^T\right]^T \quad (32b)$$

$$\tilde{\mathbf{f}} = \left[\mathbf{f}_{11}^T \ldots \mathbf{f}_{17}^T, \ldots, \mathbf{f}_{n1}^T \ldots \mathbf{f}_{n7}^T\right]^T \quad (32c)$$

Now, upon pre-multiplication of the transpose of $\tilde{\mathbf{N}}$ of (27) to (31) gives the minimal set of constrained equations of motion, i.e.,

$$\tilde{\mathbf{N}}^T\tilde{\mathbf{M}}\dot{\tilde{\mathbf{v}}} = \tilde{\mathbf{N}}^T\tilde{\mathbf{f}} \quad (33)$$

where $\dot{\tilde{\mathbf{v}}} = \tilde{\mathbf{N}}\ddot{\boldsymbol{\theta}} + \dot{\tilde{\mathbf{N}}}\dot{\boldsymbol{\theta}}$. Equations in (33) lead to the set of independent equations of motion in generalized coordinates given by

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} = \boldsymbol{\tau} \quad (34)$$

in which the $\mathbf{I}$ is $n \times n$ matrices**,** is called the Generalized Inertia Matrix (GIM) and $\mathbf{C}$ is the matrix containing convective inertia terms. Moreover, $\boldsymbol{\tau}$ is n-dimensional vector of torques. The expressions of $\mathbf{I}, \mathbf{C}$ and $\boldsymbol{\tau}$ are given by

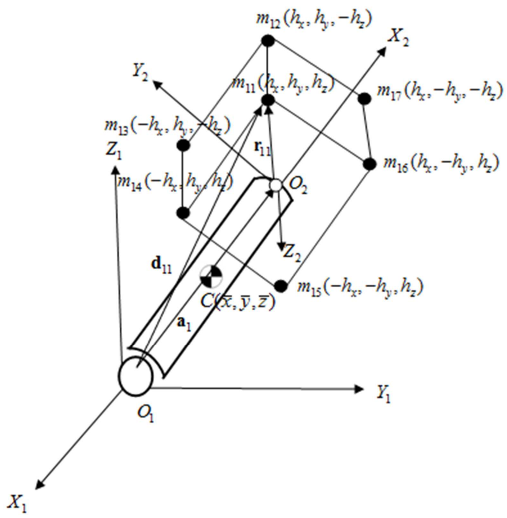


Fig. 2. Spatial pendulum with its seven point model.

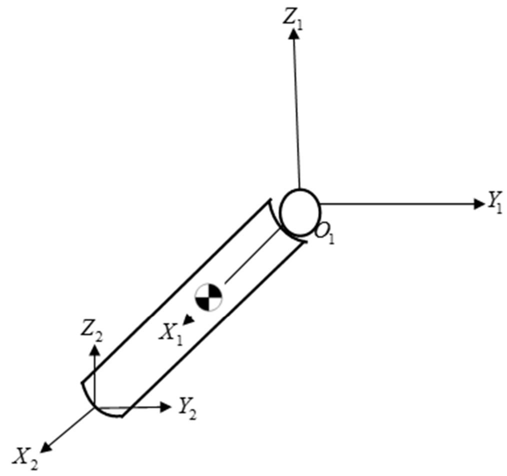


Fig. 3. Spatial pendulum at zero Euler angles

$$\mathbf{I} \equiv \tilde{\mathbf{N}}^T \tilde{\mathbf{M}} \tilde{\mathbf{N}};\ \mathbf{C} \equiv \tilde{\mathbf{N}}^T \tilde{\mathbf{M}} \dot{\tilde{\mathbf{N}}};\ \boldsymbol{\tau} = \tilde{\mathbf{N}}^T \tilde{\mathbf{f}} \qquad (35)$$

Note that the expressions in (35) provide the same set of final expressions obtained using the Newton-Euler uncoupled equations of motion and the corresponding DeNOC matrices, as derived in [11,13].

V. ILLUSTRATION: SPATIAL PENDULUM

Consider a spatial pendulum shown in Fig. 2. The rigid body of the pendulum is coupled to the fixed-base by a spherical joint which has 3-DoF. The pendulum is represented by the set of rigidly connected seven point-masses. The point-masses are located at the vertices of a parallelepiped whose center is located at origin O_2 of the body-fixed frame, as shown in Fig. 2. For the joint motion of the 3-DOF spherical joint, Euler Angles (EA) can be used. Out of 12 possible EA sets to represent a spherical joint, the ZYZ set was chosen for the purpose of dynamic modelling of the spatial pendulum using the system of point-masses. For the ZYZ set, the rotation matrix required for several coordinate transformation of matrices and vector is given from [13] as

$$\mathbf{Q} \equiv \begin{bmatrix} C\phi C\theta C\varphi - S\phi S\varphi & -C\phi C\theta S\varphi - S\phi C\varphi & C\phi S\theta \\ S\phi C\theta C\varphi + C\phi S\varphi & -S\phi C\theta S\varphi + C\phi C\varphi & S\phi S\theta \\ -S\theta C\varphi & S\theta S\varphi & C\theta \end{bmatrix} \quad (36)$$

where S and C stand for the "Sine and "Cosine", whereas ϕ, θ, φ are the ZYZ Euler angle set. Fig. 3 shows the pendulum position for zero EA set.

The coordinates of the point-masses or the component of vector $\mathbf{r}_{1j}$, for $j = 1,...,7$, calculated using (11-13) in the body-fixed frame are given in Table 1.

The mass of the point-masses were calculated using (1-7) for the following input parameters of the rigid body: m =2.512 kg, a =1m, diameter of link, d =.02m, mass center location, $\bar{x} = -a/2, \bar{y} = 0, \bar{z} = 0$, and the moment of inertia and the product of inertia, $I_{xx} = md^2/8$, $I_{yy} = I_{zz} = (md^2/16) + (ma^2/3)$, $I_{xy} = I_{yz} = I_{zx} = 0$ kg-m^2. Note that the mass-center and the inertia tensor were referred in the body-fixed frame $O_2 X_2 Y_2 Z_2$. The point-masses were then calculated as,

$m_{11} = 0.628kg$; $m_{12} = -0.543kg$; $m_{13} = 1.171kg$;
$m_{14} = 0kg$; $m_{15} = 1.171kg$; $m_{16} = -0.543kg$;
$m_{17} = 0.628kg$.

A. General Inertia Matrix

The general inertia matrix was derived using the expression given in (35) as, $\mathbf{I} \equiv \tilde{\mathbf{N}}^T \tilde{\mathbf{M}} \tilde{\mathbf{N}}$, where the 21×21 mass matrix for the seven point-mass system is given as, $\tilde{\mathbf{M}} \equiv diag.[m_{11}\mathbf{1},...,m_{17}\mathbf{1}]$ and $\tilde{\mathbf{N}} \equiv \mathbf{D}\mathbf{N}_l\mathbf{N}_d$, where the

TABLE I. COMPONENT OF VECTOR $\mathbf{r}_{1j}$ IN BODY-FIXED FRAME, I.E., $O_2 X_2 Y_2 Z_2$

	$\mathbf{r}_{11}$	$\mathbf{r}_{12}$	$\mathbf{r}_{13}$	$\mathbf{r}_{14}$	$\mathbf{r}_{15}$	$\mathbf{r}_{16}$	$\mathbf{r}_{17}$
Along X_2	h_x	h_x	$-h_x$	$-h_x$	$-h_x$	h_x	h_x
Along Y_2	h_y	h_y	h_y	h_y	$-h_y$	$-h_y$	$-h_y$
Along Z_2	h_z	$-h_z$	$-h_z$	h_z	h_z	h_z	$-h_z$

21×6 point-mass matrix $\mathbf{D}$, the 6×6 matrix $\mathbf{N}_l$, and the 6×3 matrix $\mathbf{N}_d$ for the 3-DoF spherical joint, are given by

$$\mathbf{D} \equiv \begin{bmatrix} -\mathbf{d}_{11} \times \mathbf{1} & \mathbf{1} \\ \vdots & \vdots \\ -\mathbf{d}_{17} \times \mathbf{1} & \mathbf{1} \end{bmatrix}; \mathbf{N}_l \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{1} \end{bmatrix}; \mathbf{N}_d \equiv \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{O} \end{bmatrix} \quad (37)$$

where the 3-dimensional vectors $[\mathbf{d}_{1j}]_2$, for $j = 1, \cdots, 7$ in body-fixed frame is given as

$$[\mathbf{d}_{11}]_2 \equiv \begin{bmatrix} a_1 + h_x & a_1 + h_y & a_1 + h_z \end{bmatrix}^T;$$

$$[\mathbf{d}_{12}]_2 \equiv \begin{bmatrix} a_1 + h_x & a_1 + h_y & a_1 - h_z \end{bmatrix}^T;$$

$$[\mathbf{d}_{13}]_2 \equiv \begin{bmatrix} a_1 - h_x & a_1 + h_y & a_1 - h_z \end{bmatrix}^T;$$

$$[\mathbf{d}_{14}]_2 \equiv \begin{bmatrix} a_1 - h_x & a_1 + h_y & a_1 + h_z \end{bmatrix}^T;$$

$$[\mathbf{d}_{15}]_2 \equiv \begin{bmatrix} a_1 - h_x & a_1 - h_y & a_1 + h_z \end{bmatrix}^T;$$

$$[\mathbf{d}_{16}]_2 \equiv \begin{bmatrix} a_1 + h_x & a_1 - h_y & a_1 + h_z \end{bmatrix}^T;$$

$$[\mathbf{d}_{17}]_2 \equiv \begin{bmatrix} a_1 + h_x & a_1 - h_y & a_1 - h_z \end{bmatrix}^T$$

In case these vectors need to be represented in base-frame, they can be calculated as $[\mathbf{d}_{1j}]_1 \equiv \mathbf{Q}[\mathbf{d}_{1j}]_2$, where the matrix $\mathbf{Q}$ is given by (36). The matrices $\mathbf{1}$ and $\mathbf{O}$ in (37) are the 3×3 identity and zero matrices, whereas matrix $\mathbf{L}_1$ is 3×3 matrix associated with the angular velocity of the spatial pendulum represented as $\boldsymbol{\omega}_1$ with the rates of the ZYZ Euler angles represented as $\dot{\boldsymbol{\theta}}_1 \equiv \begin{bmatrix} \dot{\phi} & \dot{\theta} & \dot{\varphi} \end{bmatrix}^T$, i.e.,

$$\boldsymbol{\omega}_1 = \mathbf{L}_1 \dot{\boldsymbol{\theta}}_1 \quad (38a)$$

where the 3×3 matrix $\mathbf{L}_1$ has the following representation [13]:

$$\mathbf{L}_1 \equiv \begin{bmatrix} 0 & -S\phi & S\theta C\phi \\ 0 & C\phi & S\theta S\phi \\ 1 & 0 & C\theta \end{bmatrix} \quad (38b)$$

B. Convective Inertia terms

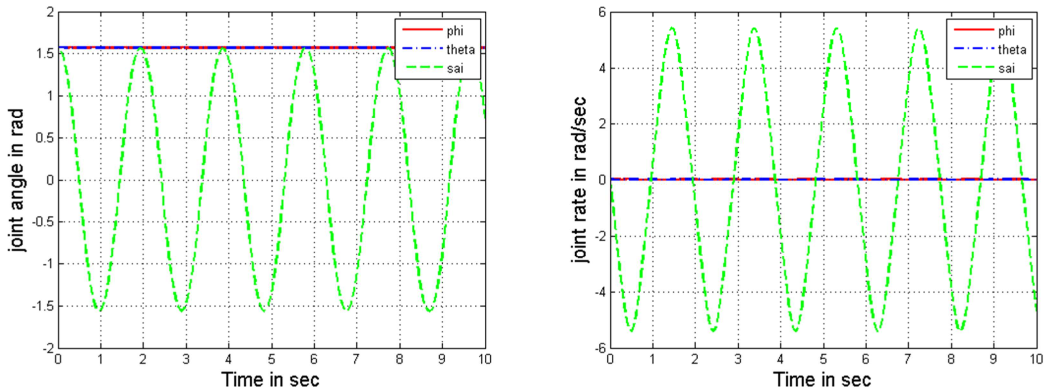
The convective inertia terms were calculated using (35) as, $\mathbf{C} \equiv \tilde{\mathbf{N}}^T \tilde{\mathbf{M}} \dot{\tilde{\mathbf{N}}}$, where, $\dot{\tilde{\mathbf{N}}} \equiv \dot{\mathbf{D}}\mathbf{N}_l\mathbf{N}_d + \mathbf{D}\dot{\mathbf{N}}_l\mathbf{N}_d + \mathbf{D}\mathbf{N}_l\dot{\mathbf{N}}_d$ in which the elements of $\dot{\mathbf{D}}$ were calculated as $[\dot{\mathbf{d}}_{1j}]_1 \equiv \dot{\mathbf{Q}}[\mathbf{d}_{1j}]_2$, and $\dot{\mathbf{N}}_l$ is the 6×6 matrix of zeros. The matrix $\dot{\mathbf{N}}_d$ has the following expression:

$$\mathbf{N}_d \equiv \begin{bmatrix} \dot{\mathbf{L}}_1 \\ \mathbf{O} \end{bmatrix} \quad (39)$$

C. Generalized forces

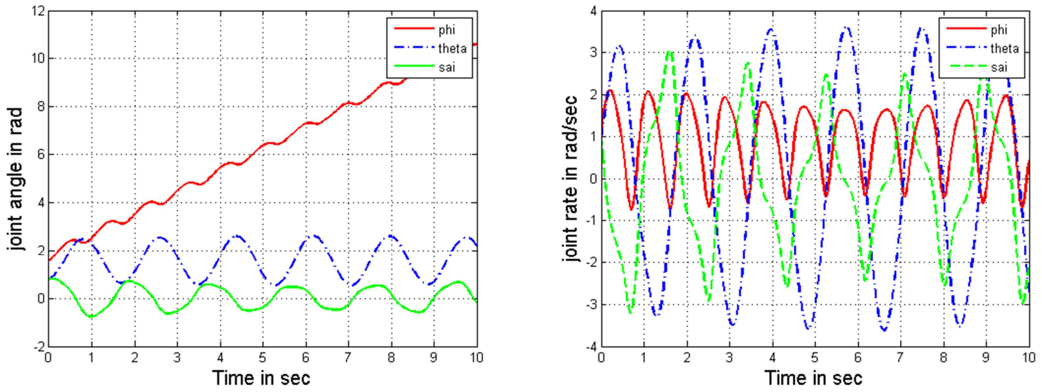
The generalized forces were calculated using (35) as $\boldsymbol{\tau} = \tilde{\mathbf{N}}^T \tilde{\mathbf{f}}$, where $\tilde{\mathbf{f}}$ includes externally applied forces and those due to gravity acting at each point-mass. The expression of force is given by $\tilde{\mathbf{f}} = \mathbf{f}^E + \mathbf{f}^G$ in which the 21-dimensional vector $\mathbf{f}^E$ and $\mathbf{f}^G$ are as follows:

$$\mathbf{f}^E \equiv [\mathbf{0}, ..., \mathbf{0}]^T \text{ and } \mathbf{f}^G \equiv [m_{11}\mathbf{g}, ..., m_{17}\mathbf{g}]^T \quad (40)$$



(a) Joint angles (b) Joint rates

Fig. 4.Simulation corresponding to planar behavior



(a) Joint angles (b) Joint rates

Fig.5. Simulation for spatial behavior

where, $\mathbf{g} \equiv [0 \quad 0 \quad -g]^T$; $\mathbf{0} = [0 \quad 0 \quad 0]^T$ Hence, $\boldsymbol{\tau} \equiv \tilde{\mathbf{N}}^T \mathbf{f}^E + \tilde{\mathbf{N}}^T \mathbf{f}^G \equiv \boldsymbol{\tau}_1 + \boldsymbol{\tau}^G$.

D. Simulation

Simulation was performed using (35) and the corresponding expressions derived in Section V. The input parameters in terms of the point-masses were calculated in the beginning of Section V. Since the free-fall simulation was obtained, no input torque, i.e., $\boldsymbol{\tau}_1 = \mathbf{0}$, was considered. The acceleration due gravity is g=9.81 m/s^2. The initial joint angles in terms of the ZYZ Euler angles were taken as $\phi = 90^\circ, \theta = 90^\circ, \varphi = 90^\circ$, whereas joint rates were taken as zeros, i.e., $\dot{\boldsymbol{\theta}}_1(0) = \mathbf{0}$. Fig. 4 shows the simulation results. Since the configuration corresponds to a planar solid pendulum, as analyzed in [13], the results show exact match, thus, apparently validating the algorithm based on the point-mass system. To study the spatial behavior, another set of initial conditions were considered, which are: $\phi = 90^\circ, \theta = 45^\circ, \varphi = 45^\circ$,whereas joint rates were taken as zeros, i.e., $\dot{\boldsymbol{\theta}}_1(0) = \mathbf{1}$ m/s. The resulting plots are shown in Fig. 5 indicating clearly spatial behavior of the pendulum.

VI. CONCLUSIONS

In this paper, dynamic modelling methodology for spatial multibody systems is presented using the equimomental system of point-masses, where each body is represented as a set of rigidly connected seven point-masses. The corresponding point-mass matrix and the DeNOC matrices were derived from the velocities of the point-masses. A numerical example is provided with a spatial pendulum where the joint motion has 3-DoF. The derived matrices were used to form the minimal set of constrained equations of motion from the unconstrained Newton's equations of linear motion. Simulation of the spatial pendulum was performed using MATLAB's "ode45" function.

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