Minimization of Preload in Springs used in Static Balancing of Linkages under Constant Loads

Gaurav Singh[1] and G. K. Ananthasuresh[2]
Dept. of Mechanical Engineering
Indian Institute of Science
Bangalore, India
[1]gauravs@mecheng.iisc.ernet.in; [2]suresh@mecheng.iisc.ernet.in

***Abstract*— Minimizing the potential energy of statically balanced linkage eases the assembly of the balancing springs and reduces the loads experienced by the constituent members. Current techniques for statically balancing a linkage using only springs and no auxiliary bodies have free parameters to be chosen by the designer. We present a technique that utilizes these free parameters to impose additional conditions that make the potential energy not just constant in all configurations but also a minimum among all possible design alternatives. The conditions required for minimum potential energy for a statically balanced lever and a statically balanced planar 2-R linkage are derived in this paper. These results are then generalized for any planar linkage by noting that the lower bound for the balanced potential energy is equal to the maximum external work among all possible configurations. Two practical examples that use Peaucellier-Lipkin and scissors linkages are included to exemplify the method.**

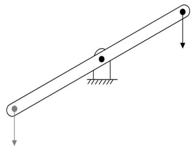
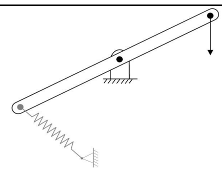
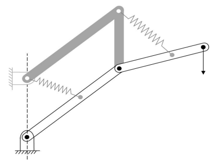
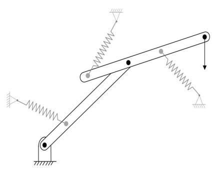
Keywords—static balancing; zero-free-length spring; preload; potential energy

I. INTRODUCTION

Static balancing of a linkage is the addition of compensating gravity loads and/or spring loads so that the linkage is in static equilibrium in all its configurations; i.e., its potential energy is the same in all the configurations of the linkage. As shown in Table 1, gravity loads acting on a linkage can be balanced by adding either gravity loads or spring loads.

Static balancing of a linkage under gravity loads by the addition of compensating gravity loads/counterweights (case (a) in Table 1) is well known and has been addressed by many researchers (e.g., [1] and [2]). A review of these techniques is given in [3]. On the other hand, as shown in case (b) in Table 1, using zero-free-length springs for balancing gravity loads is relatively new. Zero-free-length springs are different from the normal springs because they have zero length between the two connection points when the spring force is zero. Practical methods to modify normal springs into zero-free-length springs have been demonstrated by Herder [4]. Static balancing by addition of spring loads can be done with or without adding auxiliary bodies, as shown in cases (c) and (d) in Table 1. Different methods of balancing using springs, which add auxiliary bodies are presented in [5], [6], and [7]. Static balancing by adding spring loads without any auxiliary bodies is discussed in [8-11].

TABLE I. DIFFERENT METHODS TO STATICALLY BALANCE A LINKAGE UNDER GRAVITY LOADS (BALANCING GRAVITY LOADS, SPRINGS AND AUXILIARY BODIES ARE SHOWN IN GREY COLOR, WHILE ORIGINAL LOADS AND BODIES ARE SHOWN IN BLACK)

Balancing using gravity loads	Balancing using spring loads	
 (a)	 (b)	
	(c) With auxiliary bodies 	(d) Without auxiliary bodies 

Between the two options of using auxiliary bodies or not, static balancing without any auxiliary bodies is advantageous as it reduces the number of bodies and simplifies the linkage albeit adding extra springs. In general, the weight of the auxiliary body is more than that of a spring. Hence, the ideal design that does not take into account the weight of the balancing entities is likely to be closer to reality in practice if auxiliary bodies are avoided. Furthermore, the auxiliary bodies might change the kinematics of the original mechanism.

On the other hand, the disadvantage of using only springs and no auxiliary bodies is that the spring forces of the balancing springs might become very high. These high spring forces might require modification of the dimensions of the members of the linkage to withstand high spring forces. High spring forces may also require increased effort during the assembly of the balancing springs. Furthermore, the damage caused in the event of failure of any of the balancing springs will be more severe if the spring forces are high. Therefore, in this paper, we present a technique with which balancing spring loads can be kept at the theoretical minimum value. Consequently, one disadvantage of using only balancing springs is overcome as much as it is theoretically possible.

The balancing technique given in [8], [9], and [11] involves spring constants and coordinates of the anchor points of the zero-free-length springs required for the perfect static balancing of a given linkage. There are free choices among them because the number of parameters exceeds the number of equations governing the constant potential energy conditions. Thus, extra parameters can be chosen in such a way that the potential energy is not just constant but also minimum among all possible alternatives of the parameters. This enables the designer to reduce the balancing spring loads and the loads in the constituent members of the balanced linkage. First, we illustrate this with the simplest example of a pivoted crank carrying a constant load.

Static balancing of gravity load by a spring load was first proposed by Lucien Lacoste [12] for making a pendulum of enhanced time period. As shown in Fig. 1, a lever under the action of gravity load W acting at the point $\mathbf{p}=\begin{bmatrix} l & 0\end{bmatrix}^T$ in the local coordinate system of the lever is balanced by a zero-free-length spring of spring constant, k. The fixed anchor point of the spring in the global reference frame is $\mathbf{b}=\begin{bmatrix} 0 & b_Y\end{bmatrix}^T$ while the anchor point on the lever in the local reference frame of the lever is $\mathbf{a}=\begin{bmatrix} a_x & 0\end{bmatrix}^T$.

The potential energy of the spring-lever system, which includes a part that is due to the load (PE_c) and the other part that is due to the spring (PE_s), is given by

$$\begin{aligned} PE &= PE_c + PE_s \\ &= Wl\sin\theta + \frac{k}{2}\left(\left(a_x\cos\theta\right)^2 + \left(a_x\sin\theta - b_Y\right)^2\right) \\ &= \frac{k}{2}\left(a_x^2+b_Y^2\right) + \left(Wl - ka_xb_Y\right)\sin\theta \end{aligned} \tag{1}$$

For static balancing, the potential energy given by (1) must be independent of the configuration variable, θ. This can be achieved by equating the coefficient of $\sin\theta$ term to zero:

$$\begin{aligned} &Wl - ka_xb_Y = 0 \\ &\Rightarrow k = \frac{Wl}{a_xb_Y} \end{aligned} \tag{2}$$

The constant potential energy of the statically balanced system is then given by

$$PE = \frac{Wl}{2a_xb_Y}\left(a_x^2+b_Y^2\right) = \frac{Wl}{2}\left(\frac{a_x}{b_Y}+\frac{b_Y}{a_x}\right) \tag{3}$$

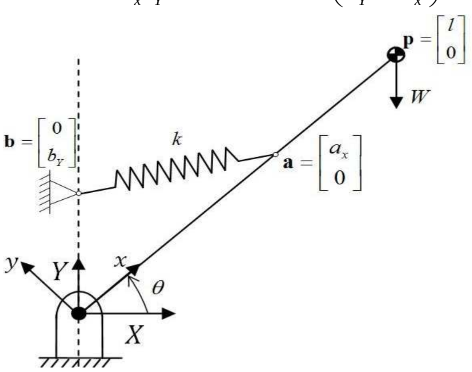


Fig. 1. Static balancing of lever under gravity load.

There are three parameters, namely, k, a_x, and b_Y, to be chosen by the designer here. But there is only one equation, i.e., (2) to ensure static balance in all configurations. Therefore, there are two free parameters. We utilize them to minimize the constant value of the potential energy of the balanced system given by (3). On partially differentiating the balanced potential energy given by (3), with respect to a_x and b_Y, we obtain the following necessary condition which defines the stationary points of PE:

$$a_x = \pm b_Y \tag{4}$$

For checking the sufficient condition, consider the Hessian of the potential energy function, $\mathbf{H}(PE)$, given by (3):

$$\mathbf{H}(PE) = \begin{bmatrix} Wl\dfrac{b_Y}{a_x^3} & \dfrac{Wl}{2}\left(\dfrac{-1}{b_Y^2}-\dfrac{1}{a_x^2}\right) \\ \dfrac{Wl}{2}\left(\dfrac{-1}{b_Y^2}-\dfrac{1}{a_x^2}\right) & Wl\dfrac{a_x}{b_Y^3} \end{bmatrix} \tag{5}$$

It may be verified that the determinant of the Hessian for both the conditions implied in (4) is zero.

$$\det(\mathbf{H}(a_x, \pm a_x)) = 0 \tag{6}$$

It may also be checked that, at the stationary point $a_x = b_Y$, the Hessian is positive semi-definite as its eigenvalues are:

$$\lambda_1 = 0 \text{ and } \lambda_2 = 4/a_x^2 \tag{7}$$

On the other hand, at the stationary point $a_x = -b_Y$, the Hessian is negative semi-definite as its eigenvalues are:

$$\lambda_1 = 0 \text{ and } \lambda_2 = -4/a_x^2 \tag{8}$$

Hence, the minimum value of the constant potential energy occurs at the critical point $a_x = b_Y$ and this minimum potential energy is given by

$$PE_{\min} = Wl \tag{9}$$

A surface plot of the potential energy given by (3) as a function of a_x and b_Y is shown in Fig. 2. We can see from Fig. 2 that the potential energy is minimum for the points corresponding to $a_x = b_Y$. This corresponds to an entire set, which is a valley of minima along the line $a_x = b_Y$ in the $a_x - b_Y$ plane.

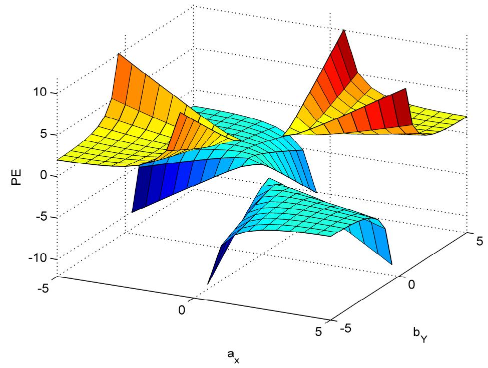


Fig. 2. Surface plot of potential energy vs. a_x and b_Y.

It may be noticed that the value of the minimum potential energy given by (9) is equal to the maximum value of the potential energy due to the gravity load among all possible configurations, i.e., $\theta = 0^0$ to 360^0. The potential energy due to the gravity load is maximum when $\theta = 90^0$. In this configuration, the strain energy is equal to zero by virtue of (4).

In fact, the result that minimum balanced potential energy being equal to maximum potential energy due to the gravity loads is not surprising. This is because making strain energy as low as possible is the aim here because potential energy due to the loads is what it is for the given mechanism dimensions and the load. The lowest possible value of the strain energy in this case happens to be zero. From the first row of (1), we get

$$PE_s = PE - PE_c \tag{10}$$

Since, strain energy is always positive, we can infer that

$$\begin{aligned} & PE - PE_c \geq 0 \\ & \Rightarrow PE \geq PE_c \end{aligned} \tag{11}$$

From (11), we can see that the minimum value of potential energy of the system is at best equal to the maximum value of the potential energy due to the gravity loads. Hence, *if the strain energy corresponds to zero in the configuration of maximum external work, then the constant potential energy obtained is the global minimum for the given linkage under gravity loads*. We use this as a guideline in the generalizations considered in this paper. The rest of the paper is organized as follows.

The method used for minimizing the preload in the spring of statically balanced linkage is presented in Section 2. The conditions for minimum potential energy for a generalized lever and a 2R linkage under gravity load are derived in this section. In Section 3, it is shown through practical examples of a four-bar linkage and a Peaucellier-Lipkin linkage that the method developed in Section 2 can be directly used to obtain the balancing spring parameters for minimum potential energy of other statically balanced linkages. Static balancing of a scissors mechanism using a zero-free-length spring and a finite-free-length spring is shown in Section 4 along with the condition for minimum potential energy. Summarization of the method and conclusions are in Section 5.

II. METHOD FOR MINIMIZING SPRING PRELOAD IN STATICALLY BALANCED LINKAGES

A. Generalized Lever

In this problem, as can be seen in Fig. 3, there are five spring parameters, namely, the spring constant, k; local coordinates of the anchor point on the lever, $\mathbf{a} = [a_x \quad a_y]^T$, and global coordinates of the fixed anchor point, $\mathbf{b} = [b_X \quad b_Y]^T$. Let the constant force with respect to the global reference frame be $\mathbf{f} = [0 \quad -W]^T$ and its point of action in the local coordinate system of the lever be $\mathbf{p} = [l \quad 0]^T$.

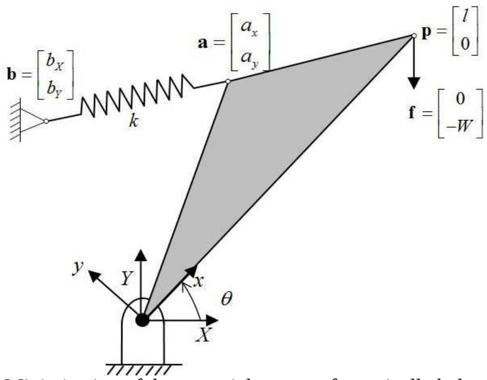


Fig. 3. Minimization of the potential energy of a statically balanced lever.

The potential energy of the system is given by

$$PE = PE_s + PE_c$$
$$= \begin{Bmatrix} \dfrac{k}{2}(a_x^2 + a_y^2 + b_X^2 + b_Y^2) \\ +k(-a_x b_X - a_y b_Y)\cos\theta \\ +k(-a_x b_Y + a_y b_X)\sin\theta \end{Bmatrix} + \{Wl\sin\theta\} \tag{12}$$

The potential energy of the system given in (12) can be written as a linear combination of the terms involving $\sin\theta$, $\cos\theta$ and 1, where θ is the angle made by the lever with the x-axis. If the coefficients of $\sin\theta$ and $\cos\theta$ terms become equal to zero, then the net potential energy becomes invariant to the configuration variable, θ. Hence, the conditions required for static balancing of the lever are given by

$$a_x b_X + a_y b_Y = 0 \tag{13}$$

$$Wl + k\left(-a_x b_Y + a_y b_X\right) = 0 \tag{14}$$

Here, we have parameters (a_x, b_Y, b_X, b_Y, and k) that define the spring and these are governed by two equations, (13) and (14). Let us express a_y and k in terms of the other three parameters using (13) and (14).

$$a_y = -\frac{a_x b_X}{b_Y} \tag{15}$$

$$k = \frac{Wl\, b_Y}{a_x\left(b_X^2 + b_Y^2\right)} \tag{16}$$

This gives us the freedom to choose a_x, b_X, and b_Y in such a way that the system has minimum potential energy. In view of (15) and (16), the potential energy of the system given by (12) can be simplified as

$$PE = \frac{Wl}{2a_x b_Y}\left(a_x^2 + b_Y^2\right) \tag{17}$$

which happens to be identical to the expression of potential energy in (3). It is worth noting that PE in (17) does not depend on b_X. Furthermore, as in (4) and the subsequent discussion, $a_x = b_y$ gives a valley of minima of PE in this case too. The minimizing condition given by

$$a_x = b_Y \tag{18}$$

transforms (15) and (16) to

$$a_y = -b_X \tag{19}$$

$$k = \frac{Wl}{\left(b_X^2 + b_Y^2\right)} \tag{20}$$

Thus, we are still left with two free choices among a_x, a_y, b_X, b_Y, and k and the remaining three can be calculated using (18) - (20).

Equations (18) and (19) can be geometrically interpreted as follows. First, it can be noted that the strain energy of this system is zero when $\theta = 90^o$. As mentioned earlier, this is also the configuration with maximum potential energy due to the constant load. Thus, at $\theta = 90^o$, the length of the spring is zero implying that the two anchor points of the spring coincide. Consequently, by transformation of the coordinates of point $\mathbf{a}$ from the local coordinate system of the lever to the global coordinate system gives the coordinates of point $\mathbf{b}$. Here, we are free to choose the coordinates of point $\mathbf{a}$, since we have two free choices. The spring constant can be determined by substituting values of $\mathbf{a}$ and $\mathbf{b}$ in the (14).

Having noted some features of the solution, we attempt to generalize this to a 2R linkage and others.

B. 2R Linkage

Consider a 2R linkage shown in Fig. 4. Quantities $\mathbf{a_i}$, $\mathbf{b_i}$, k_i, $\mathbf{f_i}$, and $\mathbf{p_i}$ denote the same quantities as in the generalized lever of Fig. 3 except that now a subscript is added to indicate the number i of the body starting from the fixed pivot. For simplification, first we consider the case where the y-coordinate of the point $\mathbf{p}_1$ in the local reference frame of body 1 to be zero, i.e., the point $\mathbf{p}_1$ lies on the line joining the fixed pivot and the joint between bodies 1 and 2. Also, θ_1 and θ_2 denote the angle of the bodies 1 and 2 with respect to the X-axis of the global coordinate system.

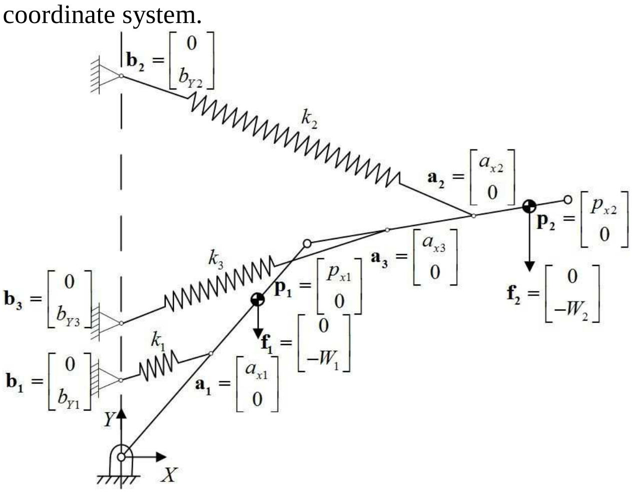


Fig. 4. Minimization of the potential energy of a simplified 2R linkage.

In this example, we need three springs for static balancing [9] and since each spring has five parameters, we have a total of 15 spring parameters to choose. For simplifying the analysis, the coordinates a_y and b_X of all the springs are taken to be equal to zero. This leaves us with a total of nine parameters to choose and the potential energy of the system is given by (21). We can obtain the conditions required for static balancing by equating the coefficients of $\sin\theta_1$, $\sin\theta_2$ and $\cos(\theta_1 - \theta_2)$ to zero. Hence, the conditions required for static balancing are given by

$$W_2 p_{x2} - k_2 a_{x2} b_{Y2} - k_3 a_{x3} b_{Y3} = 0 \tag{22}$$

$$k_2 a_{x2} + k_3 a_{x3} = 0 \tag{23}$$

$$W_1 p_{x1} - k_1 a_{x1} b_{Y1} + l\left(W_2 - k_2 b_{Y2} - k_3 b_{Y3}\right) = 0 \tag{24}$$

As already mentioned, we have nine parameters to choose which are governed by three equations, (22-24). So, we can express k_1, k_2, and k_3 in terms of the other six parameters ($a_{x1}, b_{y1}, a_{x2}, b_{y2}, a_{x3}$, and b_{y3}) using (22-24).

$$k_1 = \frac{W_1 p_{x1} + W_2 l + l\left(\frac{-W_2 p_{x2} b_{Y2}}{a_{x2}\left(b_{Y2} - b_{Y3}\right)} + \frac{W_2 p_{x2} b_{Y3}}{a_{x3}\left(b_{Y2} - b_{Y3}\right)}\right)}{a_{x1} b_{Y1}} \tag{25}$$

$$k_2 = \frac{W_2 p_{x2}}{a_{x2}\left(b_{Y2} - b_{Y3}\right)} \tag{26}$$

$$k_3 = \frac{-W_2 p_{x2}}{a_{x3}\left(b_{Y2} - b_{Y3}\right)} \tag{27}$$

In view of the (25-27), the potential energy of the statically balanced 2R linkage is given by

$$PE = \frac{W_1 p_{x1} + W_2 l + l\left(\frac{-W_2 p_{x2} b_{Y2}}{a_{x2}\left(b_{Y2} - b_{Y3}\right)} + \frac{W_2 p_{x2} b_{Y3}}{a_{x3}\left(b_{Y2} - b_{Y3}\right)}\right)}{2 a_{x1} b_{y1}}\left(a_{x1}^2 + b_{Y1}^2\right) + \frac{W_2 p_{x2}}{2 a_{x2}\left(b_{Y2} - b_{Y3}\right)}\left(a_{x2}^2 + b_{Y2}^2 + l^2\right) - \frac{W_2 p_{x2}}{2 a_{x3}\left(b_{Y2} - b_{Y3}\right)}\left(a_{x3}^2 + b_{Y3}^2 + l^2\right) \tag{28}$$

To minimize the constant value of the potential energy of the balanced system given by (28), we partially differentiate it with respect to $a_{x1}, b_{y1}, a_{x2}, b_{y2}, a_{x3}$, and b_{y3}. This gives us the necessary conditions for minimum potential energy.

$$a_{x1} = b_{Y1} \tag{29}$$

$$a_{x2} = b_{Y2} - l \tag{30}$$

$$a_{x3} = b_{Y3} - l \tag{31}$$

Substitution of the PE-minimizing conditions given by (29-31) into (25-27) gives the remaining spring parameters required for minimum potential energy.

$$PE = PE_s + PE_c = \left\{ \begin{array}{l} \frac{k_1}{2}\left(a_{x1}^2 + b_{Y1}^2\right) + \frac{k_2}{2}\left(a_{x2}^2 + b_{Y2}^2 + l^2\right) + \frac{k_3}{2}\left(a_{x3}^2 + b_{Y3}^2 + l^2\right) \\ + \left(-k_1 a_{x1} b_{Y1} - k_2 l b_{Y2} - k_3 l b_{Y3}\right)\sin\theta_1 \\ + \left(-k_2 a_{x2} b_{Y2} - k_3 a_{x3} b_{Y3}\right)\sin\theta_2 + \left(k_2 l a_{x2} + k_3 l a_{x3}\right)\cos\left(\theta_1 - \theta_2\right) \end{array} \right\} + \left\{ \begin{array}{l} \left(W_1 p_{x1} + W_2 l\right)\sin\theta_1 \\ + W_2 p_{x2}\sin\theta_2 \end{array} \right\} \tag{21}$$

$$k_2 = \frac{W_2 p_{x2}}{(b_{Y2} - l)(b_{Y2} - b_{Y3})} \tag{32}$$

$$k_3 = \frac{-W_2 p_{x2}}{(b_{Y3} - l)(b_{Y2} - b_{Y3})} \tag{33}$$

$$k_1 = \frac{W_1 p_{x1} + W_2 l + l(-k_2 b_{Y2} - k_3 b_{Y3})}{a_{x1} b_{Y1}} \tag{34}$$

In this case too, as anticipated, the minimum potential energy corresponds to zero strain energy in the configuration of maximum potential energy due to constant loads, i.e., at $\theta_1 = \theta_2 = 90°$. Thus, at $\theta_1 = \theta_2 = 90°$, the length of all the three balancing springs are equal to zero implying that $\mathbf{a_i}$ coincides with $\mathbf{b_i}$ for all the balancing springs. Hence, instead of obtaining the minimization conditions by partially differentiating (28) with respect to the free variables, we can obtain (29-31) by choosing any value for a_{x1}, a_{x2}, and a_{x3} and then at $\theta_1 = \theta_2 = 90°$ transforming the coordinates of $\mathbf{a_i}$ from the local coordinate systems of the members to the global coordinate system to obtain the coordinates of $\mathbf{b_i}$. The spring constants can be obtained by substituting values of $\mathbf{a_i}$ and $\mathbf{b_i}$ in (22-24).

Now, we drive the conditions for minimum potential energy for a 2R linkage when the centre of gravity of body 1 does not lie on the line joining the fixed pivot and the joint between bodies 1 and 2. This corresponds to a general case of 2R linkage without any simplifying assumptions as shown in Fig. 5. In this case, minimization of potential energy subjected to static balancing constraints as done for lever and simplified 2R earlier results in long expressions, which are difficult to be simplified into closed form solutions. Hence, we attach the springs in such a way that their extension is zero in the configuration of maximum external work to obtain the corresponding spring parameters, because this arrangement corresponds to global minimum for the potential energy as proved earlier.

The x-axis of the local coordinate system of the second body can be taken as along the line joining its centre of gravity and the joint with first body without any loss of generality. External work due to the gravity loads will be maximum when the second body is at 90° with respect to the global coordinate system, i.e., $\theta_2 = 90°$ and the first body is at the angle, θ_1 which is given by:

$$\theta_1 = \tan^{-1}\left(\frac{W_1 p_{x1} + W_2 l}{W_1 p_{y1}} \right) \tag{35}$$

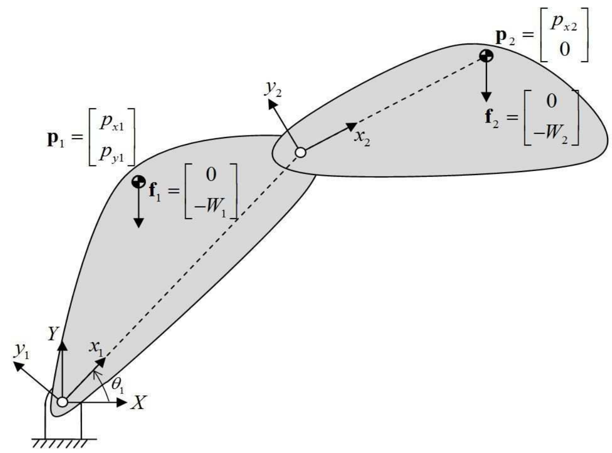


Fig. 5. General configuration of gravity loaded 2R linkage.

In this case also, we need three springs for static balancing [9] and so, we have 15 spring parameters to be determined. Let us assume the coordinate a_y of all the springs as zero. This leaves us with 12 spring parameters and the potential energy of the system is given by (36). Therefore, the conditions required for static balancing are:

$$W_2 p_{x2} - k_2 a_{x2} b_{Y2} - k_3 a_{x3} b_{Y3} = 0 \tag{37}$$

$$-k_2 a_{x2} b_{X2} - k_3 a_{x3} b_{X3} = 0 \tag{38}$$

$$k_2 a_{x2} + k_3 a_{x3} = 0 \tag{39}$$

$$W_1 p_{x1} - k_1 a_{x1} b_{Y1} + l(W_2 - k_2 b_{Y2} - k_3 b_{Y3}) = 0 \tag{40}$$

$$W_1 p_{y1} - k_1 a_{x1} b_{X1} + l(-k_2 b_{X2} - k_3 b_{X3}) = 0 \tag{41}$$

For minimum potential energy, the springs have to be attached to the 2R linkage in such a way that the strain energy of all the balancing springs are zero in the configuration of maximum external work, i.e., when $\theta_2 = 90°$ and the first body is at angle, θ_1 given by (35). This can occur only if the anchor points $\mathbf{a_i}$ coincide with $\mathbf{b_i}$ for all the springs in this configuration. Therefore, choosing any value for a_{x1}, a_{x2}, and a_{x3} and then in the configuration corresponding to θ_1 given by (35) and $\theta_2 = 90°$, transforming the coordinates of $\mathbf{a_i}$ from the local coordinate system of the bodies to the global coordinate system, we obtain the coordinates of $\mathbf{b_i}$. The coordinates of $\mathbf{b_i}$ obtained are given by

$$b_{Y1} = a_{x1} \sin\theta \tag{42}$$

$$b_{X1} = a_{x1} \cos\theta \tag{43}$$

$$b_{X2} = b_{X3} = l \cos\theta \tag{44}$$

$$PE = PE_s + PE_c = \left\{ \begin{array}{l} \frac{k_1}{2}\left(a_{x1}^2 + b_{X1}^2 + b_{Y1}^2\right) + \frac{k_2}{2}\left(a_{x2}^2 + b_{X2}^2 + b_{Y2}^2 + l^2\right) + \frac{k_3}{2}\left(a_{x3}^2 + b_{X3}^2 + b_{Y3}^2 + l^2\right) \\ +\left(-k_1 a_{x1} b_{Y1} - k_2 l b_{Y2} - k_3 l b_{Y3}\right)\sin\theta_1 + \left(-k_2 a_{x2} b_{Y2} - k_3 a_{x3} b_{Y3}\right)\sin\theta_2 \\ +\left(-k_1 a_{x1} b_{X1} - k_2 l b_{X2} - k_3 l b_{X3}\right)\cos\theta_1 + \left(-k_2 a_{x2} b_{X2} - k_3 a_{x3} b_{X3}\right)\cos\theta_2 \\ +\left(k_2 l a_{x2} + k_3 l a_{x3}\right)\cos\left(\theta_1 - \theta_2\right) \end{array} \right\} + \left\{ \begin{array}{l} \left(W_1 p_{x1} + W_2 l\right)\sin\theta_1 \\ +W_2 p_{x2} \sin\theta_2 \\ +W_1 p_{y1} \cos\theta_1 \end{array} \right\} \tag{36}$$

$$b_{Y2} = a_{x2} + l\sin\theta \tag{45}$$

$$b_{Y3} = a_{x3} + l\sin\theta \tag{46}$$

By substituting (42-46) in the static balancing equations given by (37-41), we get the spring constants of the balancing springs.

$$k_2 = \frac{W_2 p_{x2}}{a_{x2}\left(a_{x2} - a_{x3}\right)} \tag{47}$$

$$k_3 = \frac{-W_2 p_{x2}}{a_{x3}\left(a_{x2} - a_{x3}\right)} \tag{48}$$

$$k_1 = \frac{W_1 p_{y1} + \dfrac{l W_2 p_{x2} b_{X2}}{a_{x2} a_{x3}}}{a_{x1}^2 \cos\theta} \tag{49}$$

We can reduce the number of balancing springs to two, if we equate the spring constant k_1 given by (49) to zero. This will reduce the number of free variables to two because either a_{x2} or a_{x3} has to be obtained from the following equation.

$$\frac{W_1 p_{y1} + \dfrac{l W_2 p_{x2} b_{X2}}{a_{x2} a_{x3}}}{a_{x1}^2 \cos\theta} = 0 \\ \Rightarrow a_{x2} a_{x3} = \frac{-l W_2 p_{x2} b_{X2}}{W_1 p_{y1}} \tag{50}$$

For balancing a 2R linkage shown in Fig. 5 with only two springs, the balancing spring parameters have to satisfy (44) – (48) and (50).

It is shown in [9] that if a linkage consists of N bodies (without counting the ground body) and with only one body pivoted to the ground, then for static balancing, we need to attach one spring to the body pivoted to the ground and two springs to all the other bodies. Also, these springs should satisfy two equations for the body pivoted to the ground and four equations each for all the other bodies. Therefore, for a linkage consisting of N bodies, we need $(2N-1)$ balancing springs, which are governed by $(4N-2)$ balancing equations. Each spring has five parameters, therefore for $(2N-1)$ springs, we have $5(2N-1)$ parameters. For minimum potential energy of the balanced linkage, we have to impose additional conditions of zero strain energy for each spring. Therefore, further $(2N-1)$ equations will be added. Hence, for static balancing with minimum potential energy of a linkage with N bodies, the number of free choices are $(4N-2)$ $\left(=5(2N-1)-(4N-2)-(2N-1)\right)$. This implies that the designer will have $(4N-2)$ free choices even after obtaining minimum potential energy for the statically balanced linkage. This is shown in Table 2 for different values of N.

TABLE II. NUMBER OF FREE CHOICES AVAILABLE AMONG THE SPRING PARAMETERS FOR A STATICALLY BALANCED LINKAGE WITH MINIMUM POTENTIAL ENERGY

No. of bodies (N)	No. of springs ($2N-1$)	No. of spring parameters $5(2N-1)$	No. of balancing equations ($4N-2$)	Free parameters ($4N-2$)
1	1	5	2	2
2	3	15	6	6
3	5	25	10	10
4	7	35	14	14

III. PRACTICAL EXAMPLES

A. Four-bar Linkage

We can obtain the conditions for minimum potential energy for a four-bar linkage under gravity loads by relaxing one of the joint constraints and considering it as a combination of a lever and a 2R linkage. We have derived the conditions for minimum potential energy for both the lever and the 2R linkage in section 2. Combining these with the static balancing conditions, we can carry out the static balancing of the four-bar linkage with minimum potential energy. The four-bar linkage with the centers of gravity and gravity loads acting on each body is shown in Fig. 6.

In Fig. 6, W_1, W_2 and W_3 are equal to 10, 20 and 10 N respectively and p_{x1}, p_{y1}, p_{x2} and p_{x3} are 1, 0.5, 2 and 0.5 m respectively. Relaxing the revolute joint between bodies 2 and 3, we consider the bodies 1 and 2 as a 2R linkage and body 3 as a lever as shown in Fig. 7.

For balancing the 2R linkage shown in Fig. 7, we have to attach three balancing springs. The spring parameters are given by (42–49). As already mentioned in section 3, we have three free choices available in this case. Using the same notation for spring parameters as in the previous section, we assume $a_{x1} = 1$ m, $a_{x2} = 2$ m and $a_{x3} = -2$ m. The remaining spring parameters are obtained by entering the values of a_{x1}, a_{x2}, and a_{x3} in (42–49). The values obtained for the coordinates of the fixed anchor points and spring constants of the balancing springs are: $b_{X1} = 0.124$ m, $b_{Y1} = 0.9923$ m, $b_{X2} = b_{X3} = 0.1861$ m, $b_{Y2} = 3.4884$ m, $b_{Y3} = -0.5116$ m, $k_2 = 5$ N/m, $k_3 = 5$ N/m, and $k_1 = 17.81$ N/m.

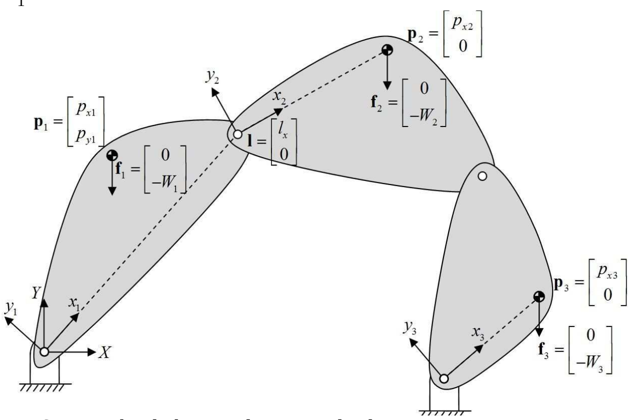


Fig. 6. Four-bar linkage under gravity loads.

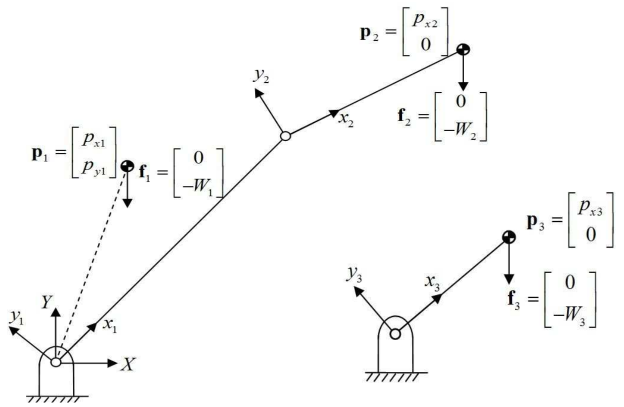


Fig. 7. Four-bar linkage broken into a 2R linkage and a lever, which are balanced separately.

Now, for balancing body 3 shown in Fig. 7, we have to attach one balancing spring and the spring parameters will be given by (18-20). In this case, we have two free choice and therefore, we assume $a_{x4} = 0.5$ m and $a_{y4} = 0$ m. The values obtained for the coordinates of the fixed anchor point and spring constant of the balancing springs are: $b_{Y4} = 0.5$ m, $b_{X4} = 0$ m and $k_4 = 20$ N/m. Hence, the spring parameters obtained above will make the potential energy of the given four-bar linkage not just constant but also a minimum among all possible such alternatives, implying the spring preloads are as low as possible.

B. Peaucellier-Lipkin Linkage

We now carry out the static balancing of a Peaucellier-Lipkin linkage as shown in Fig. 8, so that it can be used for the effort-less height adjustment of a writing-board according to the user's height. To minimize the effect of frictional forces, the Peaucellier-Lipkin linkage is used, since it can transform rotary motion into perfect straight line motion without using any linear guide ways. Two identical Peaucellier-Lipkin linkages, which are mirror images of each other, are used for the vertical motion of the writing-board. The board is attached at the joint between the bodies 6 and 7 as shown in Fig. 9. Static balancing is done for the weight of the board as well as the self-weight of each body of the linkage.

Half of the board's weight acting at the joint between the bodies 6 and 7 and the self weight of each body of the mechanism acting at its centre of gravity is shown in Fig. 8. The magnitude of the gravity loads acting on the mechanism are: $W_1 = 6.88$ N, $W_2 = W_3 = 11.43$ N, $W_4 = W_5 = 4.78$ N, $W_6 = W_7 = 7.62$ N, and $W_b = 72.03$ N. Coordinates of the centers of gravity of each body are shown in Fig. 9.

The given linkage can be statically balanced by breaking it into two 2R linkages and one 3R linkage. The bodies 1, 4, and 5 form the 3R linkage whereas the bodies 2 and 6 and bodies 3 and 7 form two 2R linkages. To reduce the number of balancing springs that have to be added, the gravity loads W_4 and W_5 acting at the midpoint of bodies 4 and 5 respectively can be transferred to the joints at both the ends and can be considered to be acting on bodies 1, 2, and 3 as shown in Fig. 10. Hence, the 3R linkage can be reduced to a lever under gravity load and can be balanced using only one spring. The two 2R linkages have the same geometry and magnitude of gravity loads. Hence, the balancing spring parameters calculated for one of them can be used for either of them.

For the balancing of body 1, we use the conditions obtained for the lever given by (18-20). Assuming $a_{x1} = 0.2$ m and $a_{y1} = 0$ m, we obtain remaining spring parameters as $b_{Y1} = 0.2$ m, $b_{X1} = 0$ m, and $k_1 = 45.88$ N/m. Now, for balancing the 2R linkage formed by the bodies 2 and 6, we use the conditions obtained for the 2R linkage in (29-34). To further reduce the number of springs, we can rearrange (24) to write

$$W_1 p_{x1} + W_2 l + l\left(-k_2 b_{Y2} - k_3 b_{Y3}\right) = k_1 a_{x1} b_{Y1} \tag{51}$$

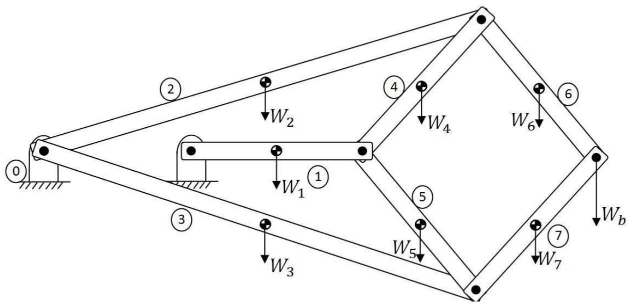


Fig. 8. Peaucellier-Lipkin linkage under gravity loads

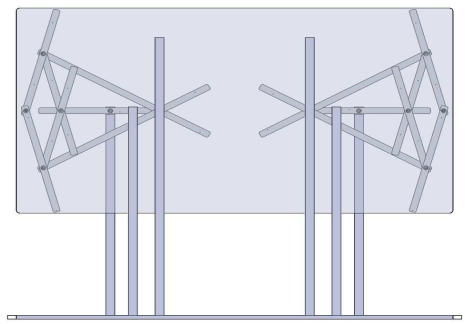


Fig. 9. Two identical Peaucellier-Lipkin linkages used for the vertical motion of a writing-board.

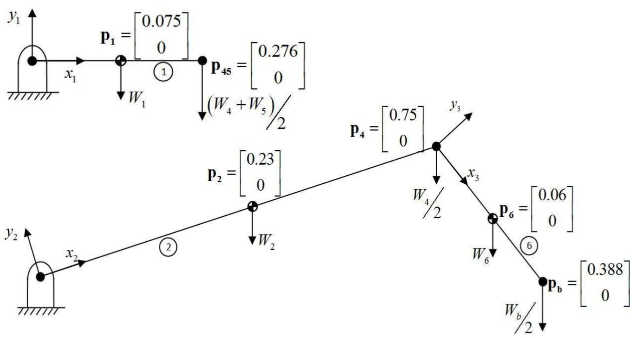


Fig. 10. Peaucellier-Lipkin linkage broken into two identical 2R linkages and a lever, which are balanced separately.

In (51), if the spring parameters are chosen such that the LHS becomes zero, then we can balance the 2R linkage using only two springs. By assuming $a_{x2} = -a_{x3}$, we obtain the remaining balancing spring parameters as $k_2 = k_3 = 33.02$ N/m, $a_{x2} = -a_{x3} = 0.468$ m, $b_{Y2} = 1.218$ m, and $b_{Y3} = 0.283$ m. As already mentioned, the spring parameters required for balancing the 2R linkage formed by the bodies 3 and 7 is the same as obtained above for the 2R linkage formed by the bodies 2 and 6.

IV. STATICALLY BALANCED SCISSORS MECHANISM

The techniques given in the current literature [4-11] use only zero-free-length springs for static balancing. We show that a scissors mechanism under gravity loads can be statically balanced using a zero-free-length spring and a finite free-length spring. The attachment of the two balancing springs is shown in Fig. 11.

Let the spring constants of the balancing springs be k_1 and k_2 and the free length be l_{10} and l_{20}. Strain energy due to the balancing springs is given by

$$\begin{aligned} PE_s &= \frac{k_1}{2}\left(\frac{3a}{2}\sin\theta - \frac{a}{2}\sin\theta - l_{10}\right)^2 \\ &+ \frac{k_2}{2}(a\cos\theta - 0 - l_{20})^2 \\ &= \frac{k_1}{2}a^2\sin^2\theta + \frac{k_2}{2}a^2\cos^2\theta - k_1 a l_{10}\sin\theta \\ &- k_2 a l_{20}\cos\theta + \frac{k_1}{2}l_{10}^2 + \frac{k_2}{2}l_{20}^2 \end{aligned} \tag{52}$$

Let the mass of each body be denoted by m_i where the subscript denotes the body i as shown in Fig. 10. Centre of mass is taken as the geometric centre of the bodies. The potential energy due to the gravity loads is given by

$$\begin{aligned} PE_C &= g\left(\begin{array}{l} m_2\frac{a}{2}\sin\theta + m_3\frac{a}{2}\sin\theta + m_5\frac{3a}{2}\sin\theta + \\ m_6\frac{3a}{2}\sin\theta + m_7 2a\sin\theta + m_8 2a\sin\theta \end{array}\right) \\ &= ga\sin\theta\left(\frac{m_2}{2} + \frac{m_3}{2} + \frac{3m_5}{2} + \frac{3m_6}{2} + 2m_7 + 2m_8\right) \end{aligned} \tag{53}$$

The potential energy expressions due to both spring loads and gravity loads ((52) and (53)) can be presented in a tabular column as shown in Table 3.

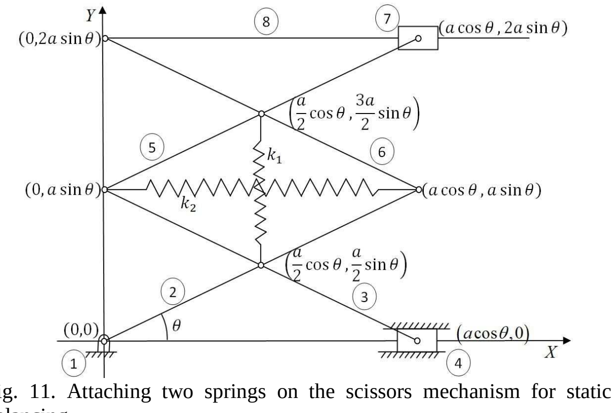


Fig. 11. Attaching two springs on the scissors mechanism for static balancing.

TABLE III. POTENTIAL ENERGY OF THE GRAVITY LOADS AND THE SPRING LOADS ACTING ON THE SCISSORS MECHANISM AS A LINEAR COMBINATION OF $\sin^2\theta$, $\cos^2\theta$, $\sin\theta$, $\cos\theta$, AND 1.

Terms	Coefficients of PE_S	Coefficients of PE_C
$\sin^2\theta$	$\frac{k_1}{2}a^2$	0
$\cos^2\theta$	$\frac{k_2}{2}a^2$	0
$\sin\theta$	$-k_1 a l_{10}$	$ga\left(\frac{m_2}{2}+\frac{m_3}{2}+\frac{3m_5}{2}+\frac{3m_6}{2}+2m_7+2m_8\right)$
$\cos\theta$	$-k_2 a l_{20}$	0
1	$\frac{k_1}{2}l_{10}^2 + \frac{k_2}{2}l_{20}^2$	0

For static balancing, the potential energy should be invariant to the configuration variable, θ. In table 3, the potential energy of both spring and gravity loads is expressed as a linear combination of $\sin^2\theta$, $\cos^2\theta$, $\sin\theta$, $\cos\theta$, and 1. Now, the potential energy becomes invariant of θ, only if the coefficients of $\sin^2\theta$, $\cos^2\theta$, $\sin\theta$, and $\cos\theta$ become equal to zero. If the spring constants of the two balancing springs, k_1 and k_2 are equal then the coefficients of $\sin^2\theta$ and $\cos^2\theta$ also become equal. Using the identity $\sin^2\theta + \cos^2\theta = 1$, $\sin^2\theta$ and $\cos^2\theta$ can be eliminated from the potential energy expression. This gives the first condition for static balancing.

$$k_1 = k_2 \tag{54}$$

For eliminating the $\sin\theta$ and $\cos\theta$ terms, their coefficients can be equated to zero, giving the remaining two conditions for static balancing.

$$\begin{aligned} &-k_1 a l_{10} + ga\left(\frac{m_2}{2} + \frac{m_3}{2} + \frac{3m_5}{2} + \frac{3m_6}{2} + 2m_7 + 2m_8\right) = 0 \\ &\Rightarrow k_1 l_{10} = g\left(\frac{m_2}{2} + \frac{m_3}{2} + \frac{3m_5}{2} + \frac{3m_6}{2} + 2m_7 + 2m_8\right) \end{aligned} \tag{55}$$

$$\begin{aligned} k_2 a l_{20} &= 0 \\ \Rightarrow l_{20} &= 0 \end{aligned} \tag{56}$$

Since, k_2 and a cannot be zero, to satisfy (56), l_{20} must be equal to zero. Hence, spring 2 has to be a zero free length spring. On the other hand, l_{10} has to be non-zero to satisfy (55). This implies, that spring 1 is a finite-free-length spring. Hence, the conditions given by (54-56) statically balance the scissors mechanism with the potential energy of the balanced system given by

$$PE = \frac{k_1}{2}a^2 + \frac{k_1}{2}l_{10}^2 = \frac{k_1}{2}(a^2 + l_{10}^2) \tag{57}$$

The product of k_1 and l_{10} as given by (44) is a constant for given gravity loads. Substituting for k_1 in terms of l_{10} in (57) gives

$$PE = \frac{c}{2l_{10}}(a^2 + l_{10}^2) \tag{58}$$

Differentiating (58) with respect to l_{10} gives the condition for minimum potential energy of the system as,

$$l_{10} = a \tag{59}$$

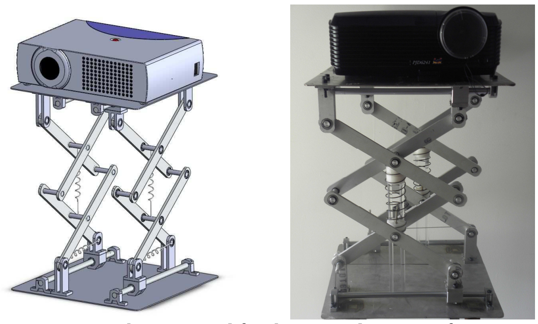


Fig. 12. Scissors mechanism used for the vertical motion of a projector.

The statically balanced scissors mechanism is used for the effortless height adjustment of a projector. Two identical scissors mechanism are used for this purpose. A platform is fixed to the body 8 of both the scissors mechanism with the projector placed on the platform as shown in Fig. 12. Static balancing is done for both the weight of the projector and self weight of all the bodies of the scissors mechanism.

The variation of potential energy of the statically balanced scissors mechanism with minimum potential energy is shown in Fig. 13. The variation of external work due to the gravity loads and strain energy due to the balancing springs is also shown.

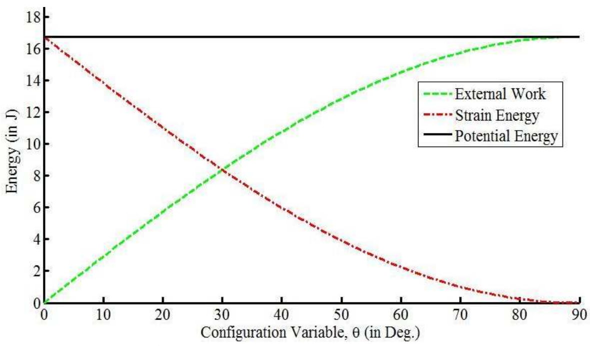


Fig. 13. Variation of the strain energy, external work, and potential energy for a statically balanced scissors mechanism.

As shown in Fig. 13, minimum potential energy corresponds to zero strain energy in the configuration of maximum external work. This is consistent with the result obtained for different linkages earlier.

V. SUMMARY AND CONCLUSIONS

By using the free parameters and other dependent balancing variables, we make the potential energy not only constant in all configurations but also a minimum among all possible design alternatives. The conditions for minimum potential energy for a statically balanced lever and a statically balanced 2R linkage are obtained. Using the conditions derived for lever and 2R linkage, minimization conditions for different mechanisms, including a four-bar linkage and Peaucellier-Lipkin linkage, are obtained and a general method is demonstrated. An important observation of this work is that the potential energy of the system that corresponds to zero strain energy at the configuration of maximum external work is the global minimum of the net potential energy for a statically balanced linkage. The main conclusion of this work is that some free choices remain even after minimizing the constant value of the potential energy of the balanced linkage. Future work will address how we can use the free choices to make the spring constants of some springs zero and thereby reduce the number of balancing springs. Also to be investigated is how the loads in the bodies of the linkage can be reduced by judicious choice of the free parameters.

REFERENCES

[1] D. Zhang, F. Gao, X. Hu, and Z. Gao, 2011, "Static Balancing and Dynamic Modeling of a Three-Degree-of-Freedom Parallel Kinematic Manipulator," IEEE International Conference on Robotics and Automation (ICRA), pp. 3211–3217.

[2] M. Jean, and C. M. Gosselin, 1996, "Static balancing of planar parallel manipulators," Proceedings of the 1996 IEEE International Conference on Robotics and Automation.

[3] G. G. Lowen, F. R. Tepper, and R. S. Berkof, 1983, "Balancing of linkages-an update," Mechanism and Machine Theory, 18(3), pp. 213-220.

[4] J. L. Herder, 2001, "Energy-free Systems: Theory, conception and design of statically balanced spring mechanisms," PhD dissertation, Delft University of Technology. ISBN 90-370-0192-0.

[5] A. Fattah, and S. K. Agrawal, 2006, "Gravity-Balancing of Classes of Industrial Robots," Proceedings of the 2006 IEEE International Conference on Robotics and Automation, May 15–19, Orlando, Florida, pp. 2872–2877.

[6] S. K. Agrawal, and A. Fattah, 2004, "Gravity-Balancing of Spatial Robotic Manipulators," Mechanism and Machine Theory, 39(12), pp. 1331–1344.

[7] T. Rahman, R. Ramanathan, R. Seliktar, and W. Harwin, 1995, "A Simple Technique to Passively Gravity-Balance Articulated Mechanisms," ASME Journal of Mechanical Design, 117(4), pp. 655–658.

[8] S. R. Deepak and G. K. Ananthasuresh, 2009, "Static Balancing of Spring-Loaded Planar Revolute-Joint Linkages Without Auxiliary Links," 14th National Conference on Machines and Mechanisms, Dec. 17–18, NIT Durgapur, India.

[9] S. R. Deepak and G. K. Ananthasuresh, 2012, "Perfect Static Balance of Linkages by Addition of Spring but not Auxiliary Bodies," ASME Journal of Mechanisms and Robotics, Vol. 4, pp. 021104-1 to 12.

[10] P. Y. Lin, W. B. Shieh, and D. Z. Chen, 2010, "Design of a Gravity-Balanced General Spatial Serial-Type Manipulator," ASME Journal of Mechanisms and Robotics, 2, 031003.

[11] S. R. Deepak, and G. K. Ananthasuresh, 2012, "Static Balancing of a Four-Bar Linkage and Its Cognates," Mechanism and Machine Theory, 48(0), pp. 62–80.

[12] L. J. B. LaCoste Jr., 1934, "A new type long period vertical seismograph," Journal of Applied Physics, 5(7), pp. 178–180.