Jacobian Based Kinematic and Static Analysis of Over-Constrained Mechanisms with Prismatic and Revolute Joints

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***Abstract*—Over-constrained and deployable mechanisms are extensively used in space and in other applications. There is an existing approach which studies the mobility and static analysis of the over-constrained and deployable mechanisms. The main feature of this approach is that the natural co-ordinates are used to define all the constraints present in the mechanisms. The constraint Jacobian matrix is developed by taking the derivatives of all the constraint equations. The null space dimension of the constraint Jacobian matrix gives the degree of freedom of the over-constrained mechanism. A numerical algorithm is used to identify the number of redundant links and joints through the constraint equations. Closed-loop kinematic solutions are found out to ensure that the over-constrained mechanism can be made deployable by actuating only one joint and all other points can be expressed in terms of this actuated variable. In this paper, the existing approach has been extended by implementing the same in an over-constrained box mechanism, where the trajectory of the joints obtained by using the constraint equations has been compared with the trajectory obtained from ADAMS. We have also extended the same approach to static analysis for an over-constrained hexagonal mechanism. The result obtained has been cross checked with that of obtained in ANSYS. Above all the new contributions of this paper is that we have used the approach for studying kinematics and statics of a mechanism having both prismatic and revolute joints which has not been done before. Secondly, the validation of the proposed theory has been done by using the above mentioned commercial packages.**

***Index Terms*—Over-constrained mechanism, Degree of freedom, Jacobian matrix.**

I. Introduction

DEGREE of freedom of a multibody system can be defined as the number of independent co-ordinates [1] required to define the mechanism. Different kind of joints used in the mechanism impose some constraints on the mechanism and decreases the mobility of the system. But many traditional method like Grübler-Kutzbach criteria [2] are based on only number of links and number of joints which determine the degree of freedom without considering the redundant constraints present in the mechanism. So, in all the over-constrained mechanism we get less number of degree of freedom than actually the system has. Again we use separate methods for studying kinematics, statics, dynamics and determining mobility of a system. There is no common method which can be used for studying all the three together. In this paper a new method based on the constraint equations of the mechanism has been used for determining mobility, studying kinematics and statics of the mechanism collectively. The associated redundant links and joints are identified through the corresponding constraint equations. Some standard package ADAMS and ANSYS has been used for cross checking the correctness of the method. In the following section the constraint Jacobian has been developed out of the constraint equations for studying the above collectively.

II. Related work

As mentioned earlier, there are many spatial mechanisms which are over-constrained but give negative degree of freedom by the traditional Grübler-Kutzback criteria even though, the mechanisms move smoothly and have typically one degree-of-freedom. In order to find out the degree of freedom, Nagaraj [1], [3] have developed a new method that studies the kinematics of such over-constrained mechanism. They have used the available constraint equations as the basis to predict the degree of freedom, obtain closed-form solutions, and obtain the static deflections of several deployable mechanisms and structures respectively. In their work, a numerical algorithm has been proposed to remove the redundant links and joints present in the over-constrained mechanisms. Kwan and Pellegrino [4] have described the stiffness of the cable used in deployable mechanism by an analytical method from which the load carrying capacity can be predicted. Although, the opening and closing are important for deployable mechanisms, the kinematics of the mechanism is also very important to discuss. Gan and Pellegrino [5] have done a systematic study of a closed-loop mechanism which can be folded into a bundle of bars. In their work, the kinematics of the deployable mechanism has been studied in detail. They have also examined the analytical and numerical solution of the loop-closure equations for such deployable mechanism. When the deployable mechanism is properly locked, it behaves as a structure that is capable of carrying load. For such structure, it is important to do the static analysis for design and practical use. Kaveh and Davaran [6] have performed the static analysis of the pantograph foldable structures. The fundamental unit of the pantograph mast is the scissor like elements. The stiffness matrix has been developed for each duplet. They have considered each link as a beam and which has three nodes. The axial deflection is taken into account but not the torsion. Aviles et al. [7] have done the kinematic analysis of the linkages through finite elements and also found out the geometric stiffness matrix. They have derived the kinematic properties of the links in a mechanism using the length constraints and the basic nodes in the links.

For a deployable mechanism it is also important to study the dynamics along with the kinematics and static analysis. A dynamic analysis of the constrained mechanical system is done by Unda and Garcia [8]. In this work, the author provides a theoretical and numerical study in both 'reference point' and 'natural' co-ordinates for dynamic analysis of the constrained mechanical system. This study presents different ways of formulating the differential equations of motions. The dynamic analysis of a planar mechanism with lower pair in basic co-ordinates is also performed in detail by Serna et al. [9]. In this work, the numerical solution for the dynamic problem of a planar mechanism has been presented. Link constraints and basic co-ordinates are taken to study the dynamic problems. Waldron [10] has nicely explained the constraint analysis of the mechanism through the constraint equations. Specifically he has focused on the kinematic study of the mechanisms through the constraint equations. Generally cables are used for opening and closing in a deployable mechanisms. Kwan and Pellegrino have studied to identify the active and passive cables [11] and how these can be expressed in terms of constraint equations, a detail analysis has been given in [12]. The literature survey done above is for the kinematic, static, and dynamic analysis [13], [14], [15] of the deployable mechanisms. The method developed by Nagaraj [1] covers the kinematic and static analysis for an over-constrained mechanism. However, the approach is limited to mechanism with revolute and spherical joints. This work is an extension of the work done by Nagaraj [1]. The main contributions of this paper are the inclusion of prismatic joints and analysis of deployable mechanisms containing combinations of prismatic, rotary joints. The other contributions of the work, such as derivation of closed-form equations, kinematic analysis using ADAMS and static analysis using ANSYS has been performed and compared with results of the approached method.

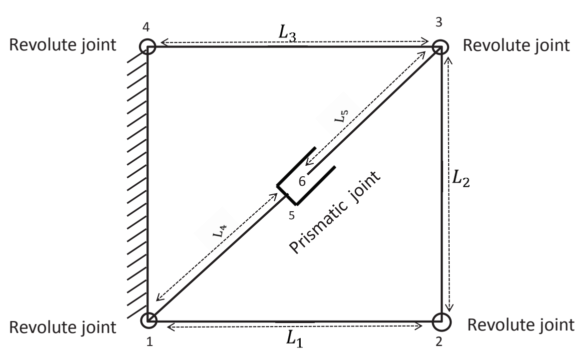

Fig. 1. A rectangular bay

A. Kinematic Analysis of Deployable Structure

The kinematic analysis of the deployable mechanism can be described using different co-ordinates like (i) relative co-ordinate (ii) reference point co-ordinates and (iii) natural co-ordinates [2]. In this paper we have described all the constraint equation in natural co-ordinates . The constraint equations are differentiated to obtain constraint Jacobian matrix([J]). In rest of the paper, we have used the Jacobian matrix ([J]) for determining the mobility and the deflection of the over-constrained mechanism and the structure respectively. An algorithm is developed to identify the redundant links and joints through their corresponding constraint equations for an over-constrained mechanism. The algorithm can be mentioned as follows.

1. Find all the joint constraints and length constraints, and then evaluate their derivatives.
2. Add all the derivatives of the constraint equations as per the following order.
   - arising out of length constraints.
   - arising out of joint constraints.
   - arising out of cable constraints.
   - arising out of boundary condition.
3. At each step, evaluate the dimension of the null space in [J]. If the dimension of the null space does not increase by adding a constraint then that constraint is considered as the redundant one.
4. Add the boundary constraint. If the dimension of the null space does not increase by adding the boundary conditions then that boundary constraint is considered as the redundant one.
5. Find the rank of the constraint Jacobian [J], which gives the mobility of the mechanism.

In order to understand the method, some example are given as follows.

B. Prismatic joint

In three dimensions, a prismatic joint allows only translational motion in one direction, restrict translational motion in other two directions and three rotations about the three axes. In this way it has five constraint equations and possesses one degree of freedom. Figure 2 shows a prismatic joint connecting two links ij and km. Here $\vec{U_c}$ is the unit vector along which the link ij moves relative to link km. As the vectors $\vec{ij}$ and $\vec{ik}$ always maintain a constant angle they produce a constraint equation. The constant value is not necessarily zero. It depends on the orientation of the two links taken to develop the prismatic joint. As per the figure 2, the constraint equations can be written as follows.

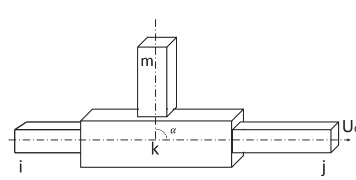

Fig. 2. Prismatic joint

$$\vec{L_{ij}} \times \vec{L_{ik}} = 0 \tag{1}$$

In addition, points i, k and j are in the same direction of the unit vector $\vec{U_c}$. Hence, we can write

$$\vec{L_{ik}} \times \vec{U_c} = 0 \tag{2}$$

Two links L_{ij} and L_{km} also maintain a constant angle throughout their motion. So it produces one more constraint equation. Equation 1 and 2 each produces two linearly independent equations and total four

$$\vec{L_{ij}} \cdot \vec{L_{km}} - L_{ij} L_{km} \cos\alpha = 0 \tag{3}$$

In this way, a prismatic joint gives five constraint equations and hence it has one degree of freedom. In two dimensions the degree of freedom of an element is three, namely two translations and one rotation. In relative co-ordinates , they are $q = (X,\ Y,\ \theta)$. When a prismatic joint is described in two dimensions, the constraint equations from figure 2 are as follows

$$(X_k - X_i)/(X_j - X_i) - (Y_k - Y_i)/(Y_j - Y_i) = 0 \tag{4}$$

$$(X_j - X_i)(X_k - X_m) + (Y_j - Y_i)(Y_k - Y_m) = L_{ij} L_{km} \cos\alpha$$

where, α is the angle between links L_{ij} and L_{km}. Revolute joint has one degree of freedom and hence there are five constraints between the two connected elements. Revolute joint can be formed when two links have a common point and a common unit vector. Here the common unit vector is $\vec{U_c}$. Both the links ij and mk are capable of rotating about the unit vector U_c. In figure 3, as the points m and j are coinciding, their co-ordinates are found to be always same. Again, the link mk always maintains a constant angle with the unit vector $\vec{U_c}$ and hence the dot product is a constant which produces a constraint. Similarly, the link ij also coincide with the unit vector $\vec{U_c}$ and hence it produces a constraint. From figure 3, the equations can be written as follows

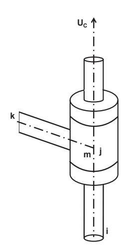

Fig. 3. Revolute joint

$$\begin{aligned} \vec{L_{mk}} \cdot \vec{U_c} - R_{mk} \cos\pi/2 &= 0 \\ \vec{L_{ij}} \cdot \vec{U_c} - R_{ij} \cos 0 &= 0 \\ X_j - X_m &= 0 \\ Y_j - Y_m &= 0 \\ Z_j - Z_m &= 0 \end{aligned} \tag{5}$$

where, R_{mk} and R_{ij} are the magnitudes of the vectors $\vec{L_{mk}}$ and $\vec{L_{ij}}$.

When the revolute joint is described in three dimensions it gives five constraint equations. However, a revolute joint gives two constraints when the motion and the connected links lie in a plane. Both are due to the common point on both the links. If the Figure 3 is described in the $X-Y$ plane then the constraint equations can be written as follows.

$$\begin{aligned} X_j - X_m &= 0 \\ Y_j - Y_m &= 0 \end{aligned} \tag{6}$$

As in two dimensions, there are three variables and two constraints, the degree of freedom is one.

C. System constraint equation

In this paper, we have considered deployable mechanisms containing only revolute and prismatic joints. In addition, we have cable constraints where the cable is modeled as a rigid link. All the constraint equations, namely those from (i) length constraints, (ii) prismatic joint constraints, (iii) revolute joints constraints, and (iv) cable constraint can be expressed collectively as follows.

$$f_j(X_1, Y_1, Z_1, X_2, Y_2, ...X_n, Y_n, Z_n) = 0 \text{ for } j = 1 \text{ to } n_c \tag{7}$$

where, n_c represents the number of constraint equations taking all the rigid links, all joint constraints and the boundary constraints. In the above equation, X_i, Y_i, and Z_i represent the natural co-ordinates required to define the length and the joint constraints. The derivatives of the system of constraint equations give the Jacobian matrix. The Jacobian matrix contains the co-efficient of the terms like $\dot{X}_1$, $\dot{Y}_1$, $\dot{Z}_1 ... \dot{X}_n$, $\dot{Y}_n$, $\dot{Z}_n$. After taking the derivative of the constraint equations, the resulting equation can be written as

$$[J]\delta X = 0 \tag{8}$$

where $[J]$ is the constraint Jacobian matrix and δX=($\dot{X}_1$, $\dot{Y}_1$, $\dot{Z}_1 ... \dot{X}_n$, $\dot{Y}_n$, $\dot{Z}_n$).

III. System description and mobility analysis by Jacobian

The typical deployable mechanisms considered here are a regular box mechanism and a regular hexagonal mechanism. Each bay is a closed loop mechanism having six links with a prismatic joint in diagonal as shown in figure 1. All the bays are identical and are connected with each other with a small link for making a closed mechanism. The constraint Jacobian method is implemented for determining mobility of a single bay. It is described as follows.

- Length constraint equation

$$\begin{aligned} (X_2 - X_1)^2 + (Y_2 - Y_1)^2 &= L_1^2 \\ (X_3 - X_2)^2 + (Y_3 - Y_2)^2 &= L_2^2 \\ (X_4 - X_3)^2 + (Y_4 - Y_3)^2 &= L_3^2 \\ (X_5 - Y_1)^2 + (Y_5 - Y_1)^2 &= L_4^2 \\ (X_6 - X_3)^2 + (Y_6 - Y_3)^2 &= L_5^2 \end{aligned} \tag{9}$$

- Prismatic constraint equation

Previously it has been mentioned that a prismatic joint has five constraint equations in three dimensions. In two dimensions the constraints are only two. The constraint equations for the prismatic joint, from figure 2, are

$$(X_6 - X_1)/(X_5 - X_1) - (Y_6 - Y_1)/(Y_5 - Y_1) = 0 \quad (10)$$
$$(X_5 - X_1)(X_6 - X_3) + (Y_5 - Y_1)(Y_6 - Y_3) = L_4 L_5$$

where, the points $P(X_1, Y_1)$,....,$P(X_6, Y_6)$ are the natural co-ordinates and L_2,...,L_6 are the link lengths.

- Boundary constraint equation

From the figure 1 it is clear that the single rectangular bay is fixed at points 1 and 4. The boundary conditions are

$$X_1 = 0, Y_1 = 0, X_4 = 0, Y_4 = L_2 \quad (11)$$

Following the algorithm presented above and putting the constraint equations step by step, the details of the Jacobian matrix analysis of the single bay are obtained. This is shown in Table 1. The degree of freedom is found to be one and this agrees with the Grübler's equations.

Constraints	Dimension of [J]	Dimension of Null Space
Length constraints	(5,12)	7
Prismatic constraints	(7,12)	5
Boundary condition ($X_1 = Y_1 = 0$)	(9,12)	3
Boundary condition ($X_4 = 0, Y_4 = L$)	(11,12)	1

TABLE I
JACOBIAN MATRIX ANALYSIS OF A SINGLE BAY

A. The box mechanism

Figure 4 shows a mechanism in the shape of a box. Each face of this box is a single bay as shown in figure 1. This mechanism is a closed-loop mechanism and also called a deployable ring mechanism. The deployment occurs by pulling the cable used in the prismatic joint. The deployment takes place in such a way that the change in length in the prismatic joint in each bay is equal at every instant. As shown in figure 1, each bay has six links, six revolute joints and one prismatic joint in the diagonal.

Constraints are considered to perform the null space analysis and for each bay the dimension of null space was evaluated. The constraint equations developed in this mechanism are added according to order: Length constraint, Prismatic joint constraint, Revolute joint constraint, Cable constraint and finally Boundary constraint. The null space analysis of the box mechanism is shown in Table II. It can be seen from Table II that a revolute joint in FACE 4 is redundant. In addition one of the cable constraints is redundant. It may be noted that if the order of the constraint equations is changed the redundancy will appear in the last face of the mechanism.

In the last column of the table II we have mentioned the number of constraint equations which bring no change in the dimension of the null space. Basically these equations correspond to the number of redundant joints and links present in the mechanism.

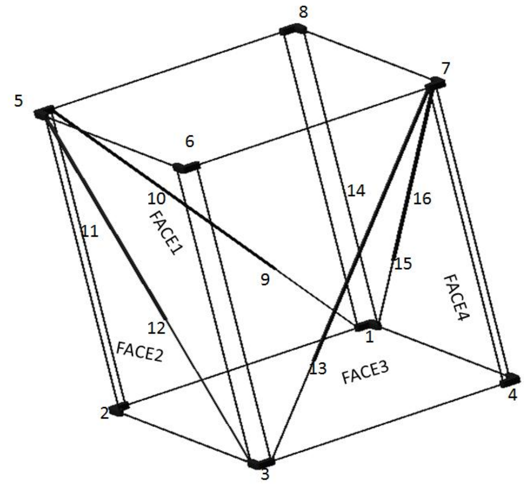

Fig. 4. Box mechanism with four bays

Constraints	Size of [J]	Dimension of Null space	Redundant components
Length Constraints	(20,48)	28	-
Prismatic constraints	-	-	-
FACE1	(23, 48)	25	-
FACE2	(26, 48)	22	-
FACE3	(29, 48)	21	2
FACE4	(32, 48)	20	2
Re volute Joint constraint	-	-	-
FACE1	(36, 48)	16	-
FACE2	(40, 48)	12	-
FACE3	(44, 48)	8	-
FACE4	(48, 48)	8	4
Cable constraint	(51, 58)	6	1
boundary constraint ($X_1 = Y_1 = Z_1 = 0$)	(54,48)	3	-
boundary constraint ($X_4 = 0, Y_4 = L, Z_4 = 0$)	(57,48)	1	-

TABLE II
JACOBIAN MATRIX FOR BOX

B. The hexagonal mechanism

A hexagonal mechanism with the six bays is shown in figure 5. The same steps as for the box mast case is used for the null space analysis. One prismatic joint constraint is found to be redundant in the prismatic joints in two faces. One revolute joint and one cable constraint is also found to be redundant. In the last step when the boundary conditions are added, the dimension of the null space reduces to 1. The degree of freedom of the hexagonal closed-loop mechanism is found to be 1. In the basic way it can be said that the number of columns of the matrix discussed in table III are number of independent variables associated with the mechanism. In each step we have taken the constraint equations of a particular type i.e. joint constraint or length constraint and simultaneously the number of redundant links and joints have been identified through the redundant constraint equations. The Jacobian matrix operation has been done in MATLAB.

Constraints	Size of [J]	Dimension of Null space	Redundant Components
Length constraints	(30, 72)	42	-
Prismatic constraints	-	-	-
FACE1	(33, 72)	39	-
FACE2	(36, 72)	38	2
FACE3	(39, 72)	36	-
FACE4	(42, 72)	33	-
FACE5	(45, 72)	30	-
FACE6	(48, 72)	30	2
Re volute joint constraint	-	-	-
FACE1	(52, 72)	26	-
FACE2	(56, 72)	22	-
FACE3	(60, 72)	18	-
FACE4	(64, 72)	14	-
FACE5	(68, 72)	10	-
FACE6	(72, 72)	10	4
Cable constraint	-	-	-
FACE1	(73, 72)	9	-
FACE2	(74, 72)	8	-
FACE3	(75, 72)	7	-
FACE4	(76, 72)	6	-
FACE5	(76, 72)	6	1
Boundary constraint $(X_1 = Y_1 = Z_1 = 0,)$ $(X_2 = Z_2 = 0, Y_2 = L)$	(82, 72)	1	-

TABLE III
JACOBIAN MATRIX FOR HEXAGON

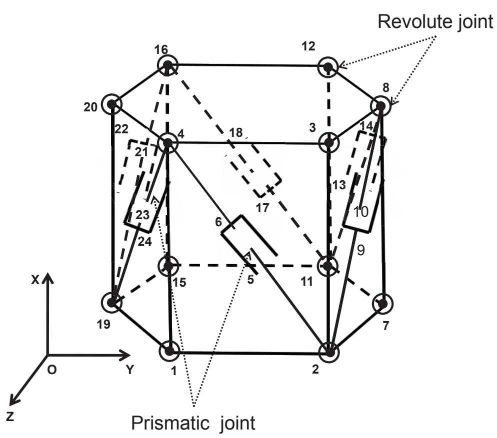

Fig. 5. A hexagonal mechanism with six bays

IV. CLOSED-FORM KINEMATIC SOLUTION OF BOX MECHANISM

A closed-form equation for a one degree-of-freedom mechanism involves expressing all joint variables in terms of one actuated joint variable. For over-constrained mechanism it is not clear how to derive these expressions since some of the joints and links are redundant and should be removed to determine these expressions. The numerical algorithm presented in the previous section indicates the redundant variables and hence in determining the closed-form expression. In this section, we present the closed-form kinematic expressions for the box-shaped deployable mechanism. These expressions have been derived in their simplest form by making use of the symbolic manipulation software MAPLE. For the box mechanism shown in figure 4, all the links are considered to be of same length. The length constraint equations are

$$(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 = L^2$$
$$(X_1 - X_{10})^2 + (Y_1 - Y_{10})^2 + (Z_1 - Z_{10})^2 = L^2$$

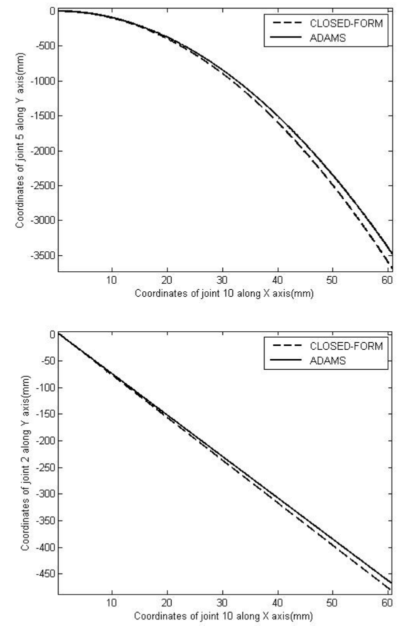

Fig. 6. Comparative plot of closed-form solution and ADAMS

$$\begin{aligned}(X_2 - X_5)^2 + (Y_2 - Y_5)^2 + (Z_2 - Z_5)^2 = L^2\\(X_5 - X_8)^2 + (Y_5 - Y_8)^2 + (Z_5 - Z_8)^2 = L^2\\(X_5 - X_9)^2 + (Y_5 - Y_9)^2 + (Z_5 - Z_9)^2 = L^2\\(X_5 - X_6)^2 + (Y_5 - Y_6)^2 + (Z_5 - Z_6)^2 = L^2\\(X_2 - X_3)^2 + (Y_2 - Y_3)^2 + (Z_2 - Z_3)^2 = L^2\\(X_3 - X_6)^2 + (Y_3 - Y_6)^2 + (Z_3 - Z_6)^2 = L^2\\(X_5 - X_{12})^2 + (Y_5 - Y_{12})^2 + (Z_5 - Z_{12})^2 = L^2\\(X_3 - X_{11})^2 + (Y_3 - Y_{11})^2 + (Z_3 - Z_{11})^2 = L^2\end{aligned} \tag{12}$$

Here, we have taken rigid link length constraint equations for two sides of the whole box. In the same way the length constraint of the other links in other two sides can be taken. The prismatic joint constraint equations are obtained using the equations (1) and (2) as

$$\begin{aligned}\frac{X_{10} - X_1}{X_5 - X_1} = \frac{Y_{10} - Y_1}{Y_5 - Y_1} = \frac{Z_{10} - Y_1}{Z_5 - Y_1}\\(X_1 - X_{10})(X_9 - X_5) + (Y_1 - Y_{10})(Y_9 - Y_5) +\\(Z_1 - Z_{10})(Z_9 - Z_{10}) = L^2\end{aligned} \tag{13}$$

The actuating joint variable is taken as P(X_{10},Y_{10},Z_{10}) and the input is chosen as Y_{10}. Taking the boundary conditions, i.e., X_1=Y_1=Z_1=0, the closed-form solutions for three arbitrary chosen points are found out as

For $(X_2, Y_2, Z_2)^T$**:**

$$X_2 = \frac{3L^2 - 4Y_{10} - 4LY_{10}}{4}$$
$$Y_2 = \frac{2Y_{10} - L}{2}$$
$$Z_2 = 0$$

For $(X_5, Y_5, Z_5)^T$**:**

$$X_5 = \frac{3L^2 - 4Y_{10}^2}{4}$$
$$Y_5 = \frac{2Y_{10} + L}{2}$$
$$Z_5 = 0$$

For $(X_9, Y_9, Z_9)^T$**:**

$$X_9 = \frac{L}{2}\sqrt{2L^2 - 2Y_{10} - L}$$

$$Y_9 = \frac{l}{4}\sqrt{\frac{L(L^2 - 2Y_{10} - L)(3L + 2Y_{10})}{3L - 2Y_{10}}}$$

$$Z_9 = 0$$

In the same way other joints can be expressed in terms of the actuating joint. The same model with following specification has been done in ADAMS.

- In the present prototype of the box mechanism, the horizontal, vertical, and the diagonal lengths of the links are taken as 175cm. As a single link, at a particular point, cannot rotate about two mutually perpendicular axes at same time, a small link has been built to connect other links by revolute joints at each corner of the box mechanism. The length and width of the small link are taken as 2.5cm and 1cm respectively.
- The two diagonal links are connected by prismatic joint and the vertical and horizontal links are connected with the revolute joints along with the small links. A user defined constraint is given in such a way that the difference between the displacements in each prismatic joint is zero.
- A linear motion, as a function of time, has been imposed in the actuating joint 10 and simulation is run for 3 second in the kinematic mode of ADAMS.

Figure 6 shows the trajectory of the joint 2 and 5 with respect to the input at the actuated joint (Y_{10}). The solid line represents the trajectory obtained from ADAMS and the dotted line represents the lines obtained from the closed-form kinematic solution of the box mechanism. It can be seen that the trajectory obtained from ADAMS has a little deviation from that of the closed-form solution. The little deviation is due to the small link present in every corner of the box mechanism. As described above that it is not possible to put two rotational joints at a single point and rotate about two different axes simultaneously. To avoid this problem a small link has been put between two links of two conjugate faces. We have taken the co-ordinates of each corner of the mechanism for evaluating the closed-form kinematic solution and length of the small link has been neglected. But in ADAMS the length of the small links have been taken into account. For this reason when we plot the trajectory of any corner joint with respect to a prismatic joint is not matching with that of the ADAMS. So the deviation is only due to the length of the small link incorporated in ADAMS.

V. Static analysis by constraint Jacobian

In this section, the stiffness matrix due to the length constraint, prismatic joint constraint, cable constraint, and stiffness due to bending has been evaluated. The stiffness due to each constraint is assembled one by one to get the equivalent stiffness matrix. Finally, the displacement of the whole mechanism has been found out in the direction of X, Y and Z, respectively. All the links considered here are assumed to have the same cross sectional area and are equal in length. The axial displacement occurs along the neutral axis of the links and all the links are considered as beams with the rotary and shear effects neglected. The Euler-Bernoulli beam equation is used.

A. Stiffness matrix for length segment

First the elongation in the axial direction is considered. From the length constraint equation, the elongation in each structural member, δL can be related to the system displacement, δX. The equation can be written as

$$[J_m]\delta X = \delta L \tag{14}$$

where, the Jacobian matrix $[J_m]$ can be written from the constraint equations. The matrix $[J_m]$ can be written as

$$\left[\begin{array}{cccccccc} (X_1 - X_5)/L_1 & .. & .. & .. & .. & .. & .. & ... \\ .. & .. & .. & .. & .. & .. & .. & ... \\ .. & .. & .. & .. & .. & .. & .. & ... \\ .. & .. & .. & .. & .. & .. & .. & ... \\ .. & .. & .. & .. & .. & .. & .. & ... \\ .. & .. & .. & .. & .. & .. & .. & ... \end{array}\right.$$

$$\left.\begin{array}{cccccccccccc} ... & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 \\ ... & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 \\ ... & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 \\ ... & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 \\ ... & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 \\ ... & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & (Z_1 - Z_4)/L_6 \end{array}\right]$$

with δX=$(\delta X_1, \delta Y_1, \delta Z_1, ...\delta X_6, \delta Y_6, \delta Z_6)^T$ and δL=$(\delta L_1, \delta L_2 ... \delta L_6)^T$. The member stiffness matrix, $[S_m]$, elongation δL and member force δT is related as

$$[S_m]\delta L = \delta T \tag{15}$$

For axial deflection the member stiffness matrix can be written as

$$[S_m] = \begin{bmatrix} A_1E_1/L_1 & .. & .. & .. & .. & 0 \\ 0 & .. & .. & .. & .. & 0 \\ 0 & .. & .. & .. & .. & 0 \\ 0 & .. & .. & .. & .. & 0 \\ 0 & .. & .. & .. & .. & 0 \\ 0 & .. & .. & .. & .. & A_6E_6/L_6 \end{bmatrix}$$

In the above expression E_i is the Young's modulus, A_i is the cross-sectional area and L_i is the length of the link i, and the member forces in each link is denoted δT=$(\delta T_1, \delta T_2, ..., \delta T_6)$. The equilibrium matrix can be represented with the terms of member forces and external load as

$$[J_m]^T \delta T = \delta F \tag{16}$$

and the applied load can be represented in its different components as

$$\delta F = (\delta F_{1x}, \delta F_{1y}, \delta F_{1z} .... \delta F_{6x}, \delta F_{6y}, \delta F_{6z})^T \tag{17}$$

Substituting the values of δT and then δL in the above equation, we get

$$[J_m]^T [S_m][J_m]\delta X = \delta F \tag{18}$$

The above equation (18) can be expressed as

$$[K_m]\delta X = \delta F \tag{19}$$

where, $[K_m] = [J_m]^T[S_m][J_m]$ represents the elastic stiffness matrix of six links in one bay.

B. Stiffness matrix due to pure bending

To derive the stiffness matrix due to bending, we consider that each link has three nodes. The bending angle can be found out by taking the cross-product of two vectors obtained from each line. In prismatic joint the bending angle can be obtained by taking the cross-product of two links which are responsible to produce the joint itself. The system displacement and the bending angle can be related by a Jacobian matrix as

$$\begin{aligned} [J_{12}]\delta X_{12} &= \delta\phi_{12} \\ [J_{23}]\delta X_{23} &= \delta\phi_{23} \\ [J_{34}]\delta X_{34} &= \delta\phi_{34} \\ [J_{14}]\delta X_{14} &= \delta\phi_{14} \\ [J_{15}]\delta X_{15} &= \delta\phi_{15} \\ [J_{63}]\delta X_{63} &= \delta\phi_{63} \end{aligned} \tag{20}$$

where, $[J_{12}]$, $[J_{23}]$, $[J_{34}]$, $[J_{14}]$, $[J_{15}]$, and $[J_{63}]$ are the Jacobian matrices, δX_{12}, δX_{23}, δX_{34},δX_{14},δX_{15},δX_{63} are the vectors representing the nodal displacement at each node. It may be noted that δX_{12}= $(\delta X_1, \delta Y_1, \delta Z_1, \delta X_m, \delta Y_m, \delta Z_m, \delta X_2, \delta Y_2, \delta Z_2)^T$ and all other variables δX_{23}, δX_{34} etc can be expressed in a similar manner. Denoting the bending angles in the global co-ordinate system by $\delta\phi_{12}$, $\delta\phi_{23}$, $\delta\phi_{34}$, $\delta\phi_{14}$, $\delta\phi_{15}$, $\delta\phi_{63}$, the transformation between the global and local quantities can be written in terms of the rotation matrices and we can write

$$\begin{aligned}
\delta\phi_{12}^{'} &= [R_{12}]\delta\phi_{12} \\
\delta\phi_{23}^{'} &= [R_{23}]\delta\phi_{23} \\
\delta\phi_{34}^{'} &= [R_{34}]\delta\phi_{34} \\
\delta\phi_{14}^{'} &= [R_{14}]\delta\phi_{14} \\
\delta\phi_{15}^{'} &= [R_{15}]\delta\phi_{15} \\
\delta\phi_{63}^{'} &= [R_{63}]\delta\phi_{63}
\end{aligned} \tag{21}$$

where, $\delta\phi_{12}^{'}....\delta\phi_{63}^{'}$ is the bending angles in the local co-ordinates . The general form of the rotation matrix in the above equation, denoted by $[R_{mn}]$, can be obtained as a product of three rotations, ψ, ϕ and θ about Y, Z and X axis respectively. The simple rotation matrices are given R_{ψ}= $\begin{pmatrix} \frac{C_x}{\sqrt{C_x^2+C_z^2}} & 0 & \frac{C_z}{\sqrt{C_x^2+C_z^2}} \\ 0 & 1 & 0 \\ \frac{-C_z}{\sqrt{C_x^2+C_z^2}} & 0 & \frac{C_x}{\sqrt{C_x^2+C_z^2}} \end{pmatrix}$,
R_{ϕ}= $\begin{pmatrix} \sqrt{C_x^2+C_z^2} & C_y & 0 \\ -C_y & \sqrt{C_x^2+C_z^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
R_{θ}= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta, \end{pmatrix}$
where, for the two end nodes m and n of the link, $C_x = \frac{X_m - X_n}{L}$, $C_y = \frac{Y_m - Y_n}{L_L}$ and $C_z = \frac{X_m - Z_n}{L}$. Using equations (20) in equations (21), we get the followings

$$\begin{aligned}
\delta\phi_{12}^{'} &= [R_{12}][J_{12}]\delta X_{12} \\
\delta\phi_{23}^{'} &= [R_{23}][J_{23}]\delta X_{23} \\
\delta\phi_{34}^{'} &= [R_{34}][J_{34}]\delta X_{34} \\
\delta\phi_{14}^{'} &= [R_{14}][J_{14}]\delta X_{14} \\
\delta\phi_{15}^{'} &= [R_{15}][J_{15}]\delta X_{15} \\
\delta\phi_{63}^{'} &= [R_{63}][J_{63}]\delta X_{63}
\end{aligned} \tag{22}$$

Considering the bending deformation and neglecting the torsion, equation (22) can be written as

$$\begin{aligned}
\delta\phi_{12}^{''} &= [J_1]\delta X_{12} \\
\delta\phi_{23}^{''} &= [J_2]\delta X_{23} \\
\delta\phi_{34}^{''} &= [J_3]\delta X_{34} \\
\delta\phi_{14}^{''} &= [J_4]\delta X_{14} \\
\delta\phi_{15}^{''} &= [J_5]\delta X_{15} \\
\delta\phi_{63}^{''} &= [J_6]\delta X_{63}
\end{aligned} \tag{23}$$

The relation between the forces and the moments are given by

$$\delta F = [J_n]^T \delta M^{''} \tag{24}$$

If the rotation $\delta\phi^{''}$ in all the links are elastic, the members moments $\delta M^{''}$ can be expressed with a diagonal matrix of member stiffness as

$$[S_n]\delta\phi^{''} = \delta M^{''} \tag{25}$$

with $\delta M^{''}$ = $\delta M_{12}^{''}, ..., \delta M_{63}^{''})^T$ and the stiffness matrix is given as $[S_m] =$

$$\begin{bmatrix}
\frac{E_1 I_z}{l_1} & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & \frac{E_1 I_y}{l_1} & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & 0 & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & .. & .. & .. & .. & .. & .. & . & .. & .. & 0 & 0 \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & 0 & 0 \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & \frac{E_6 I_z}{l_6} & 0 \\
0 & .. & .. & .. & .. & .. & .. & .. & .. & .. & .. & \frac{E_6 I_y}{l_6}
\end{bmatrix}$$

In the above equation E is the Young's modulus and I_z and I_y are the area moment of inertia of the cross section about Z and Y axes, respectively. The diagonal elements are the bending stiffness of all the links in one bay. As the deflection is due to axial tension, the elastic stiffness matrix relating to displacement and force can be written as

$$[J_n]^T[S_n][J_n]\delta X = \delta F \tag{26}$$

and this equation can be written as

$$[K_n]\delta X = \delta F \tag{27}$$

where, $[K_n]$=$[J_n]^T[S_n][J_n]$.

C. Stiffness matrix due to cable

Generally the cable is added to increase the stiffness in the structure. In the tight condition the cables are assumed as bars. The stiffness added due the cables can be taken into account in the whole stiffness. We have considered two end points of the cable to write the stiffness matrix. The matrix can be written as follow. $[K_c]$=$A_c E_c / l_c$

$$\begin{pmatrix}
r^2 & rs & rt & -r^2 & -rs & -rt \\
rs & s^2 & st & -rs & -s^2 & -st \\
rt & st & t^2 & -rt & -s^2 & -st \\
-r^2 & -rs & -rt & r^2 & rs & rt \\
-rs & -s^2 & -st & rs & s^2 & st \\
-rt & -st & -t^2 & rt & st & t^2
\end{pmatrix}$$

where, r=$\frac{X_i - X_j}{l_c}$, s=$\frac{Y_i - Y_j}{l_c}$, t=$\frac{Z_i - Z_j}{l_c}$, and A_c and E_c are the cross-sectional area and the Young's modulus of the cable, respectively. By combining the stiffness matrix due to length constraint, pure bending and cables, the elastic stiffness can be written as

$$[K_s]\delta X = \delta F \tag{28}$$

where, $[K_s]$=$[K_m]$+$[K_n]$+$[K_c]$. An hexagonal over-constrained mechanism, shown in figure 5, is taken as an example to evaluate its stiffness. Each links have the same cross-sectional area. The horizontal links have length 260mm; the vertical links have the length 400mm and the diagonal links have the 400mm. The co-ordinate values of the corner points hexagonal mast is given in Table IV. The cross-sectional area of the links and the cable is chosen as 200mm^2 and 5mm^2, respectively. The value of the Young's modulus is assumed to be $207\ \text{kN/mm}^2$ for the links and $63\ \text{kN/mm}^2$ for the cables. The second moment of inertia of the links in hexagonal structure about Y and Z axis, I_y and Iz are obtained as 6666mm^4 and 1666mm^4, respectively from the chosen geometry and material properties. When the unit load is applied at the joint 2, the stiffness calculated by the Lagrangian method is 37N/mm in the X-direction, 56.3N/mm in Y-direction and 78.2N/mm in the Z-direction, respectively. From the Lagrangian method the deflection at the point where load is applied is 0.027mm in X-direction, 0.018mm in Y- direction and 0.013mm in Z- direction. The same model has also been done in ANSYS for cross checking the correctness of the method. We have taken the exact dimensions of the links as well as the material properties in ANSYS environment. The values obtained in ANSYS are 0.025mm in X-direction, 0.016mm in Y- direction and 0.009mm in Z- direction respectively. Hence it is cleared that the Jacobian based constraint method is applicable for static analysis of any over-constrained mechanisms.

Joints	X-co-ordinates (mm)	Y-co-ordinates (mm)	Z-co-ordinates (mm)
1	0	0	0
2	260	0	0
3	390	225.16	0
4	260	450.33	0
5	0	45033	0
6	-130	-225.16	0
7	0	0	400
8	260	0	400
9	390	225.16	400
10	260	450.33	400
11	0	450.33	400
12	-130	225.16	400

TABLE IV
CO-ORDINATE VALUES FOR HEXAGONAL MAST

VI. CONCLUSION

The mobility and kinematics analysis of the mechanisms have been done basing on the constraint Jacobian matrix. From the Jacobian matrix the redundant links and joints has been identified through their corresponding constraint equations. It is observed that the trajectory obtained from the closed-form equation is matching with the trajectory obtained in ADAMS. The static analysis has also been performed by using the constraint Jacobian matrix. The results obtained through the constraint Jacobian method have been cross checked by making the same model in ANSYS and results are coming very close to each other. So the Jacobian constraint approach is applicable to all types over-constrained mechanism for studying their mobility, kinematics and statics together.

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